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PERFORMANCE OF HYPOTHESIS TESTS FOR GOMPERTZ DISTRIBUTION WITH RIGHT AND INTERVAL CENSORED DATA

(Prestasi Ujian Hipotesis bagi Taburan Gompertz dengan Data Tertapis Kanan dan Selang)

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ABSTRACT

This study compared the performance of two hypothesis tests for the parameters of the Gompertz distribution in the presence of a covariate, right and interval censored data. Firstly, the performance of maximum likelihood estimation (with and without midpoint imputation) was assessed for this model at various censoring proportion (cp), sample sizes (n) and study periods (k) by computing the values of bias, standard error (SE) and root mean square error ($RMSE$) via simulation study. Following that, the power analysis was conducted to evaluate the performance of the Wald and Likelihood ratio (LR) test for the parameters of this model at various cp, n, k and effect sizes. The results indicated that the maximum likelihood estimates obtained via midpoint imputation performed better than the ones obtained without imputation. The results of the power analysis showed that the LR test performed better for parameter β_1 whereas the Wald test performed better for parameter γ. Finally, the model was fit to the real survival data of 94 patients with breast cancer, whose lifetimes were either right or interval-censored. The covariate this study was the treatment type which were radiation therapy alone or the combination of radiation with chemotherapy.

Keywords: Gompertz; covariate; interval censored

ABSTRAK

Kajian ini membandingkan prestasi dua ujian hipotesis untuk parameter taburan Gompertz dengan adanya kovariat, data tertapis kanan dan selang. Pertama, prestasi penganggaran kebolehjadian maksimum (dengan dan tanpa imputasi titik tengah) dinilai untuk model ini pada pelbagai perkadaran penapisan (cp), saiz sampel (n) dan tempoh kajian (k) dengan mengira nilai bias, ralat piawai (SE) dan punca min kuasa dua ralat ($RMSE$) melalui kajian simulasi. Berikutan itu, analisis kuasa dikendalikan untuk menilai prestasi ujian Wald and Likelihood ratio (LR) untuk parameter model ini pada pelbagai ukuran cp, n, k dan saiz kesan. Hasilnya menunjukkan bahawa penganggaran kebolehjadian maksimum yang diperolehi melalui imputasi titik tengah berprestasi lebih baik daripada yang diperoleh tanpa imputasi. Hasil analisis kuasa menunjukkan bahawa ujian LR menunjukkan prestasi yang lebih baik untuk parameter β_1 sedangkan ujian Wald menunjukkan prestasi yang lebih baik untuk parameter γ. Akhirnya, model ini dipadankan kepada data mandirian sebenar 94 pesakit dengan barah payudara, dengan jangka hayat mereka sama ada ditapis kanan atau ditapis selang. Kovariat kajian ini ialah jenis rawatan iaitu terapi sinaran sahaja atau gabungan sinaran dengan kemoterapi.

Kata kunci: Gompertz; covariate; right-censored; interval censored; midpoint

1. Introduction

Survival analysis is the field in statistics that studies the procedures for analyzing lifetime data. The lifetime, represented by the variable T, is a continuous non-negative random variable. Kiani and Arasan (2012) stated that survival analysis mainly focuses on describing the distribution of survival time and examining how survival time relates to specific covariates. A hypothesis test is an important process in statistics. It is used to test a claim or hypothesis about a parameter in a population by using data measured in a sample. Hypothesis testing involves the construction of two statement, the null hypothesis, H_0 and alternative hypothesis, H_1.

The failure rate of the Gompertz model increases exponentially over time. Johnson *et al.* (1972) provided a comprehensive overview of the properties of the Gompertz distribution. Additionally, the survival data has the potential to be positively skewed due to the presence of incomplete data which is known as censored observations. Right censoring occurs when the study ends before the subject experience the event of interest or the subject leaves the study before the study ends. Interval censoring occurs when the event of interest is observed within some time interval, $t_i \epsilon (t_{Li}, t_{Ri}]$, where t_{Li} and t_{Ri} is the starting point and ending point of the time interval and the exact time of the event is not known. Besides that, covariates are additional elements that affect the outcome variable, *T*.

The challenge of dealing with interval censored data is prevalent in medical research, where patient inspections occur at irregular time intervals. Therefore, it is important to identify the estimation procedure that yields the best results. There is a scarcity of research discussing hypothesis testing for the Gompertz model with covariate and interval-censored data. The objective of this research is to compare the performance of two estimation techniques for the Gompertz model with covariate in the presences of right and interval censored data, namely, with and without imputation. Following that, performances of the Wald and LR hypothesis tests were compared for the parameters of this model via power analysis at different censoring proportions (*cp*), sample sizes (*n*), study periods (*k*), significance levels (α) and effect sizes.

This study only focuses on single fixed covariates that are usually measured at the beginning of the study and remains constant throughout the study period such as gender and blood group. Many authors have studied parametric models with the inclusion of covariates and censored data. Prentice (1973) carried out analysis of survival times, censored and uncensored, arising from a hazard function that varies exponentially with covariates. Modelling cure rates using the Gompertz model with fixed covariate were illustrated by Gieser *et al.* (1998) and parametric survival models for interval censored data with both fixed and time dependent covariates was presented by Sparling *et al.* (2006).

Following that, Kiani and Arasan (2012) proposed a simulation method of interval censored data in medical and biological studies and Kiani *et al.* (2012) compared several confidence interval estimation techniques for the parameters of Gompertz model with time dependent covariates in presences of right-censored data. Kiani and Arasan (2013), evaluated the performance of the Gompertz model with time-dependent covariates in the presence of interval-, right-, and left-censored data. More recently, Al-Hakeem *et al.* (2023) explored the generalized exponential distribution with interval-censored data and time dependent covariate.

Lindsey (1998) studied parametric regression models with interval censored data by using two methods, the exact likelihood and mid-point imputation. He found that the midpoint likelihood function provided good results for both measuring fit and for parameter estimation. This statement may not be reliable under all circumstances and each distribution must be examined in detail. This study focuses on mid-point imputation. Other works involving the Gompertz distribution with covariate and interval-censored data was by Naslina *et al.* (2020). However, the researchers only focused on assessing the goodness of fit using the Cox-Snell residuals and its modifications.

Al-Hakeem *et al.* (2022) obtained the parameter estimates for the generalized exponential distribution in the presence of interval censored data and covariate using a single random imputation technique. Arasan and Midi (2023) proposed a multiple random imputation method to deal with the problem of interval-censored data. Alharbi *et al.* (2022) assessed the performance of the generalized exponential model in the presence of the interval censored data with covariate using the midpoint imputation. Xin and Arasan (2024) compared of several imputation techniques, namely right, left, and mid-point imputation for log logistic model with covariate and interval censored data.

2. Methodology

Gompertz distribution was initially developed to describe the mortality pattern in human aging. However, over time, researchers began to extensively utilize it as a growth model, particularly in biology and economics, see (Winsor 1932). The probability density function (PDF) of the Gompertz distribution is given in Eq. (1),

$$f(t,\gamma,\lambda) = \lambda\, exp(\gamma t) \times exp\left[\frac{\lambda}{\gamma}(1-e^{\gamma t})\right], t \geq 0, \lambda > 0, \gamma > 0, \tag{1}$$

where, λ is known as baseline mortality, and γ is the senescent component. The corresponding survivor distribution function (SDF) and hazard function is given by Eq. (2) and Eq. (3),

$$S(t,\gamma,\lambda) = exp\left[\frac{\lambda}{\gamma}(1-e^{\gamma t})\right], \tag{2}$$

$$h(t,\gamma,\lambda) = \lambda\, exp(\gamma t). \tag{3}$$

In this research, we extend the Gompertz distribution to accommodate covariate, right and interval censored data. The effect of covariate on the survival time of the i^{th} subject can be incorporated to the hazard function by letting $\lambda = \exp(\beta' X)$. The covariate vector is denoted as $X' = (x_0, x_1, \ldots, x_p)$, where $x_0 = 1$ and, $\beta' = (\beta_0, \beta_1, \ldots, \beta_p)$ represents the unknown parameter vector, where p is the number of covariates. If we only considered a single covariate, the hazard function for i^{th} subject is given in Eq. (4),

$$h(t_i, x_i, \beta, \gamma) = exp(\beta_0 + \beta_1 x_i + \gamma t_i), \tag{4}$$

after incorporating $\lambda = e^{\beta_0 + \beta_1 x_i}$ into Eq. (3), where t_i is the survival time and x_i is the covariate value for the i^{th} subject. In this study, the maximum likelihood estimation (MLE) is employed to obtain the parameter estimates of the model. To deal with the problem of interval censored data, two approaches were used, which were (i) MLE method without midpoint imputation and (ii) MLE method with midpoint imputation, to assess which technique yields better results. Mid-point imputation techniques are used to approximate survival time by taking the midpoint of the interval $(t_{Li}, t_{Ri}]$, where $t_{Li} < t_{Ri}$ as shown in Eq. (5),

$$t_i = \frac{t_{Li} + t_{Ri}}{2} \tag{5}$$

Let δ_{E_i}, δ_{R_i} and δ_{I_i} represent indicator variables used to determine whether an observation is uncensored, right-censored, or interval-censored for the i^{th} subject, respectively. Thus,

$$\delta_{E_i} = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ subject is uncensored,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\delta_{R_i} = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ subject is right censored,} \\ 0 & \text{otherwise.} \end{cases}$$
$$\delta_{I_i} = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ subject is interval censored,} \\ 0 & \text{otherwise.} \end{cases}$$

The likelihood function for $i = 1, 2, \ldots, n$, observations without midpoint imputation is given by Eq. (6),

$$L(\beta;\gamma) = \prod_{i=1}^{n}\left[f\left(t_i, x_{i,}\beta,\gamma\right)\right]^{\delta_{Ei}}\left[S\left(r_i, x_{i,}\beta,\gamma\right)\right]^{\delta_{Ri}}\left[F\left(r_i, x_{i,}\beta,\gamma\right) - F\left(l_i, x_{i,}\beta,\gamma\right)\right]^{\delta_{Ii}}$$

$$= \prod_{i=1}^{n}\left[exp\left[\beta_0 + \beta_1 x_i + \gamma t_i + \frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right]\right]^{\delta_{Ei}}\left[exp\left[\frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right]\right]^{\delta_{Ri}}$$

$$\left[exp\left[\frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma l_i}\right)}{\gamma}\right] - exp\left[\frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma r_i}\right)}{\gamma}\right]\right]^{\delta_{Ii}}, l_i < r_i, \tag{6}$$

and log-likelihood function is shown in Eq. (7),

$$l = \sum_{i=1}^{n} \delta_{Ei}\left[\beta_0 + \beta_1 x_i + \gamma t_i + \frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right] + \delta_{Ri}\left[\frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right]$$

$$+\delta_{Ii}\left[\frac{e^{\beta_0+\beta_1 x_i}\left(e^{\gamma r_i}-e^{\gamma l_i}\right)}{\gamma}\right]. \tag{7}$$

The likelihood function for $i = 1, 2, \ldots, n,$ observations with using midpoint imputation is given in Eq. (8),

$$L_{MID}(\beta;\gamma) = \prod_{i=1}^{n}\left[f\left(t_i, x_{i,}\beta,\gamma\right)\right]^{\delta_{Ei}}\left[S\left(r_i, x_{i,}\beta,\gamma\right)\right]^{\delta_{Ri}}\left[f\left(\tilde{t}_i, x_{i,}\beta,\gamma\right)\right]^{\delta_{Ii}},\ \tilde{t}_i = \frac{l_i+r_i}{2},$$
$$\prod_{i=1}^{n}\left[exp\left[\beta_0 + \beta_1 x_i + \gamma t_i + \frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right]\right]^{\delta_{Ei}}\left[exp\left[\frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right]\right]^{\delta_{Ri}}$$
$$\left[exp\left[\beta_0 + \beta_1 x_i + \gamma \tilde{t}_i + \frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma \tilde{t}_i}\right)}{\gamma}\right]\right]^{\delta_{Ii}} \tag{8}$$

and log-likelihood function is shown in Eq. (9),

$$l_{MID} = \sum_{i=1}^{n} \delta_{Ei}\left[\beta_0 + \beta_1 x_i + \gamma t_i + \frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right] + \delta_{Ri}\left[\frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma t_i}\right)}{\gamma}\right]$$

$$+\delta_{Ii}\left[\beta_0 + \beta_1 x_i + \gamma \tilde{t}_i + \frac{e^{\beta_0+\beta_1 x_i}\left(1-e^{\gamma \tilde{t}_i}\right)}{\gamma}\right]. \tag{9}$$

The Newton Raphson (NR) iterative procedure was employed to solve the non-linear likelihood equations simultaneously. Let θ be the parameter, $\hat{\theta}$ the MLE, and θ_0 the parameter value of interest. To test whether the parameter θ is equal to a known value of interest θ_0, the null hypothesis is,

$$H_0: \theta = \theta_0. \tag{10}$$

In this research, the Wald, see Wald (1943) and Likelihood ratio (LR), see Neyman and Pearson (1928), hypothesis tests were compared via a power study. The Wald statistics is given by,

$$z = \frac{\hat{\theta} - \theta_0}{\sqrt{i^{-1}(\hat{\theta})}} \sim N(0,1) \tag{11}$$

The inverse observed information matrix, $i^{-1}(\hat{\theta})$, also known as negative Hessian provides the estimates of the variance from the diagonal of matrix evaluated at $\hat{\theta}$. The null hypothesis is rejected when test statistics, z exceed the predetermined critical value $z\alpha_{/2}$.

The likelihood ratio statistics is given by,

$$\psi = -2\left[l(\theta_0) - l\left(\hat{\theta}\right)\right]. \tag{12}$$

Power is the probability of rejecting the null hypothesis when it is false and can be obtained as follows,

$$\text{Power} = 1 - Pr(\text{ Type II Error}) = Pr(\text{ Reject } H_0 | H_0 \text{ False}). \tag{13}$$

Good hypothesis testing minimizes the probability of making errors. Therefore, power also means the probability of not committing Type II error, which occurs when we fail to reject the null hypothesis when the true parameter value is, in fact, different to the specified value. Empirical power is obtained by calculating power using simulated data. As the empirical power increases, the probability of making Type II error decreases. Two simulation studies were conducted to (i) assess the performance of MLE methods (with and without midpoint imputation) and (ii) compare and select the best between Wald and LR tests based on power analysis for the parameter of this model.

3. Simulation Study and Results

The simulation study was conducted using R software at various censoring proportion (cp), sample size (n) and study periods (k) with N = 2000 replications. The approximate censoring proportion, cp = 0.0, 0.1, 0.2, 0.3 and 0.4, sample sizes, n = 30, 50, 100, 150 and 200 and study periods, k = 12, 24 and 36 months were employed in this study. One covariate was considered, and the initial value chosen for the three parameters were β_0 = -3.5, β_1 = 0.5 and γ = 0.03. These values were selected to mimic actual survival times in months, for a more realistic simulation study. A sequence of random numbers from uniform distribution, $u_i \sim unif(0,1)$ was generated to produce,

$$t_i = \frac{\beta_0 + \beta_1 x_i - In[\gamma In(u_i)]}{\gamma}. \tag{14}$$

The covariate, x_i values were simulated from standard normal distribution, $x_i \sim N(0,1)$. To generate right and interval censored data, we firstly use an indicator, v_i, which are random variates generated from the Bernoulli distribution, with probability of success p. Thus, if p = 0.4, then 40% of the data was selected to be either interval or right censored and 60% of the data remained uncensored. Then, a set of time intervals of four months was generated according to the length of the study periods, k. Following that, observations that were chosen to be censored will be compared against these intervals where $m = 1,2, \ldots, k$ and (t_{Lm}, t_{Rm}) is the m^{th} time interval. If t_i falls in the interval (t_{Lm}, t_{Rm}) and $m \leq k$, then the observed value for t_i will be modified to (t_{Lm}, t_{Rm}). Otherwise, if $m > k$, then t_i will be right censored at t_{Rm}. The parameter estimation procedure will be assessed via the values of the bias, standard error (SE) and root mean square error $(RMSE)$.

Tables 1 to 3 display the results of the simulation study. The results clearly shows that the bias, SE and $RMSE$ for $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\gamma}$ based on both MLE methods (with and without midpoint imputation) become smaller as the sample size, (n) and study period, (k) increase and censoring proportions, (cp) decrease. Increase in the number of censored observations tend to pull the estimated values of the parameters away from actual value. Smaller values of bias, SE and $RMSE$ will indicate a superior method of estimation with reasonable accuracy and efficiency. The values of bias, SE and $RMSE$ for $\hat{\beta}_0$ and $\hat{\beta}_1$ based with midpoint imputation were slightly lower compared to the ones obtained using without midpoint imputation, when data was censored. For $\hat{\gamma}$, the value of bias, with midpoint imputation was slightly higher in comparison to the ones obtained without midpoint imputation. However, the values of SE and $RMSE$ with midpoint imputation was still generally lower compared to without midpoint imputation. Hence, we conclude that the performance of estimated based on midpoint imputation was better than the ones obtained without midpoint imputation.

Table 1: Bias, SE and $RMSE$ for $\hat{\beta}_0$ based on MLE method without midpoint imputation (with midpoint imputation is in bold)

n	cp	k = 12			k = 24			k = 36		
		bias	**SE**	**RMSE**	**bias**	**SE**	**RMSE**	**bias**	**SE**	**RMSE**
30	0.0	0.0497	0.2433	0.2483	0.0497	0.2433	0.2483	0.0497	0.2433	0.2483
		0.0497	**0.2433**	**0.2483**	**0.0497**	**0.2433**	**0.2483**	**0.0497**	**0.2433**	**0.2483**
	0.1	0.0519	0.2538	0.2590	0.0510	0.2462	0.2514	0.0499	0.2434	0.2485
		0.0516	**0.2533**	**0.2585**	**0.0507**	**0.2457**	**0.2508**	**0.0496**	**0.2429**	**0.2479**
	0.2	0.0547	0.2658	0.2714	0.0524	0.2504	0.2558	0.0511	0.2451	0.2504
		0.0543	**0.2651**	**0.2706**	**0.0518**	**0.2496**	**0.2549**	**0.0496**	**0.2429**	**0.2479**
	0.3	0.0585	0.2745	0.2807	0.0523	0.2540	0.2594	0.0505	0.2465	0.2516
		0.0578	**0.2735**	**0.2795**	**0.0514**	**0.2529**	**0.2581**	**0.0496**	**0.2455**	**0.2505**
	0.4	0.0609	0.2866	0.2930	0.0541	0.2580	0.2636	0.0506	0.2480	0.2531
		0.0599	**0.2851**	**0.2913**	**0.0531**	**0.2566**	**0.2620**	**0.0494**	**0.2468**	**0.2517**
50	0.0	0.0318	0.1740	0.1768	0.0318	0.1740	0.1768	0.0318	0.1740	0.1768
		0.0318	**0.1740**	**0.1768**	**0.0318**	**0.1740**	**0.1768**	**0.0318**	**0.1740**	**0.1768**
	0.1	0.0275	0.1842	0.1862	0.0269	0.1808	0.1828	0.0272	0.1785	0.1806
		0.0274	**0.1841**	**0.1861**	**0.0268**	**0.1807**	**0.1826**	**0.0270**	**0.1784**	**0.1804**
	0.2	0.0287	0.1887	0.1909	0.0282	0.1834	0.1855	0.0277	0.1796	0.1818
		0.0285	**0.1885**	**0.1906**	**0.0279**	**0.1831**	**0.1852**	**0.0273**	**0.1794**	**0.1814**
	0.3	0.0304	0.1986	0.2009	0.0288	0.1872	0.1894	0.0279	0.1807	0.1828
		0.0299	**0.1981**	**0.2003**	**0.0283**	**0.1867**	**0.1888**	**0.0272**	**0.1802**	**0.1822**
	0.4	0.0323	0.2113	0.2138	0.0286	0.1912	0.1934	0.0277	0.1819	0.1840
		0.0317	**0.2103**	**0.2126**	**0.0279**	**0.1905**	**0.1925**	**0.0269**	**0.1812**	**0.1832**

Table 1 (*Continued*)

100	0.0	0.0182	0.1173	0.1187	0.0182	0.1173	0.1187	0.0182	0.1173	0.1187
		0.0182	**0.1173**	**0.1187**	**0.0182**	**0.1173**	**0.1187**	**0.0182**	**0.1173**	**0.1187**
	0.1	0.0155	0.1196	0.1207	0.0153	0.1168	0.1178	0.0155	0.1162	0.1173
		0.0155	**0.1196**	**0.1206**	**0.0152**	**0.1168**	**0.1178**	**0.0153**	**0.1162**	**0.1172**
	0.2	0.0163	0.1248	0.1259	0.0157	0.1198	0.1208	0.0150	0.1180	0.1190
		0.0161	**0.1247**	**0.1257**	**0.0155**	**0.1196**	**0.1206**	**0.0147**	**0.1178**	**0.1187**
	0.3	0.0173	0.1307	0.1318	0.0162	0.1227	0.1237	0.0150	0.1191	0.1200
		0.0170	**0.1304**	**0.1316**	**0.0158**	**0.1224**	**0.1234**	**0.0146**	**0.1188**	**0.1197**
	0.4	0.0180	0.1368	0.1379	0.0164	0.1255	0.1266	0.0150	0.1202	0.1211
		0.0177	**0.1364**	**0.1375**	**0.0159**	**0.1251**	**0.1261**	**0.0143**	**0.1198**	**0.1207**
150	0.0	0.0088	0.0935	0.0939	0.0088	0.0935	0.0939	0.0088	0.0935	0.0939
		0.0088	**0.0935**	**0.0939**	**0.0088**	**0.0935**	**0.0939**	**0.0088**	**0.0935**	**0.0939**
	0.1	0.0064	0.0943	0.0945	0.0067	0.0931	0.0933	0.0060	0.0921	0.0923
		0.0063	**0.0943**	**0.0945**	**0.0066**	**0.0930**	**0.0932**	**0.0059**	**0.0921**	**0.0922**
	0.2	0.0069	0.0981	0.0984	0.0064	0.0945	0.0947	0.0060	0.0926	0.0928
		0.0068	**0.0980**	**0.0983**	**0.0062**	**0.0944**	**0.0946**	**0.0057**	**0.0924**	**0.0926**
	0.3	0.0071	0.1016	0.1019	0.0065	0.0962	0.0965	0.0061	0.0937	0.0939
		0.0070	**0.1015**	**0.1017**	**0.0062**	**0.0961**	**0.0963**	**0.0057**	**0.0935**	**0.0937**
	0.4	0.0078	0.1047	0.1050	0.0067	0.0980	0.0982	0.0060	0.0945	0.0946
		0.0075	**0.1045**	**0.1047**	**0.0018**	**0.0066**	**0.0068**	**0.0054**	**0.0942**	**0.0944**
200	0.0	0.0090	0.0803	0.0809	0.0090	0.0803	0.0809	0.0090	0.0803	0.0809
		0.0090	**0.0803**	**0.0809**	**0.0090**	**0.0803**	**0.0809**	**0.0090**	**0.0803**	**0.0809**
	0.1	0.0088	0.0832	0.0837	0.0084	0.0821	0.0825	0.0084	0.0811	0.0815
		0.0087	**0.0832**	**0.0836**	**0.0083**	**0.0820**	**0.0824**	**0.0082**	**0.0811**	**0.0815**
	0.2	0.0091	0.0854	0.0859	0.0084	0.0829	0.0833	0.0084	0.0814	0.0819
		0.0090	**0.0853**	**0.0858**	**0.0082**	**0.0828**	**0.0832**	**0.0081**	**0.0813**	**0.0817**
	0.3	0.0095	0.0878	0.0883	0.0084	0.0840	0.0844	0.0080	0.0818	0.0822
		0.0093	**0.0877**	**0.0882**	**0.0080**	**0.0838**	**0.0842**	**0.0076**	**0.0816**	**0.0820**
	0.4	0.0096	0.0912	0.0917	0.0084	0.0854	0.0858	0.0080	0.0826	0.0830
		0.0094	**0.0910**	**0.0915**	**0.0079**	**0.0852**	**0.0856**	**0.0074**	**0.0823**	**0.0827**

Table 2: Bias, SE and $RMSE$ for $\hat{\beta}_1$ based on MLE method without midpoint imputation (with midpoint imputation is in bold)

n	cp	k = 12			k = 24			k = 36		
		bias	**SE**	**RMSE**	**bias**	**SE**	**RMSE**	**bias**	**SE**	**RMSE**
30	0.0	-0.1266	0.3555	0.3773	-0.1266	0.2433	0.2483	0.0497	0.2433	0.2483
		-0.1266	**0.3555**	**0.3773**	**-0.1266**	**0.2433**	**0.2483**	**0.0497**	**0.2433**	**0.2483**
	0.1	-0.1326	0.3633	0.3867	-0.1300	0.2462	0.2514	0.0499	0.2434	0.2485
		-0.1328	**0.3629**	**0.3864**	**-0.1305**	**0.2457**	**0.2508**	**0.0496**	**0.2429**	**0.2479**
	0.2	-0.1412	0.3728	0.3986	-0.1366	0.2504	0.2558	0.0511	0.2451	0.2504
		-0.1417	**0.3721**	**0.3981**	**-0.1375**	**0.2496**	**0.2549**	**0.0496**	**0.2429**	**0.2479**
	0.3	-0.1539	0.3846	0.4142	-0.1436	0.2540	0.2594	0.0505	0.2465	0.2516
		-0.1545	**0.3834**	**0.4134**	**-0.1451**	**0.2529**	**0.2581**	**0.0496**	**0.2455**	**0.2505**
	0.4	-0.1656	0.3955	0.4287	-0.1525	0.2580	0.2636	0.0506	0.2480	0.2531
		-0.1664	**0.3938**	**0.4275**	**-0.1544**	**0.2566**	**0.2620**	**0.0494**	**0.2468**	**0.2517**
50	0.0	-0.0672	0.2546	0.2634	-0.0672	0.1740	0.1768	0.0318	0.1740	0.1768
		-0.0672	**0.2546**	**0.2634**	**-0.0672**	**0.1740**	**0.1768**	**0.0318**	**0.1740**	**0.1768**
	0.1	-0.0683	0.2677	0.2763	-0.0659	0.1808	0.1828	0.0272	0.1785	0.1806
		-0.0686	**0.2675**	**0.2761**	**-0.0664**	**0.1807**	**0.1826**	**0.0270**	**0.1784**	**0.1804**
	0.2	-0.0718	0.2732	0.2825	-0.0696	0.1834	0.1855	0.0277	0.1796	0.1818
		-0.0724	**0.2728**	**0.2822**	**-0.0706**	**0.1831**	**0.1852**	**0.0273**	**0.1794**	**0.1814**

Table 2 (*Continued*)

50	0.3	-0.0793	0.2808	0.2918	-0.0762	0.1872	0.1894	0.0279	0.1807	0.1828
		-0.0801	**0.2800**	**0.2913**	**-0.0778**	**0.1867**	**0.1888**	**0.0272**	**0.1802**	**0.1822**
	0.4	-0.0885	0.2898	0.3030	-0.0811	0.1912	0.1934	0.0277	0.1819	0.1840
		-0.0896	**0.2887**	**0.3023**	**-0.0833**	**0.1905**	**0.1925**	**0.0269**	**0.1812**	**0.1832**
100	0.0	-0.0369	0.1819	0.1856	-0.0369	0.1173	0.1187	0.0182	0.1173	0.1187
		-0.0369	**0.1819**	**0.1856**	**-0.0369**	**0.1173**	**0.1187**	**0.0182**	**0.1173**	**0.1187**
	0.1	-0.0422	0.1839	0.1887	-0.0419	0.1168	0.1178	0.0155	0.1162	0.1173
		-0.0424	**0.1838**	**0.1886**	**-0.0423**	**0.1168**	**0.1178**	**0.0153**	**0.1162**	**0.1172**
	0.2	-0.0448	0.1872	0.1925	-0.0440	0.1198	0.1208	0.0150	0.1180	0.1190
		-0.0453	**0.1869**	**0.1923**	**-0.0450**	**0.1196**	**0.1206**	**0.0147**	**0.1178**	**0.1187**
	0.3	-0.0471	0.1923	0.1980	-0.0455	0.1227	0.1237	0.0150	0.1191	0.1200
		-0.0479	**0.1918**	**0.1977**	**-0.0470**	**0.1224**	**0.1234**	**0.0146**	**0.1188**	**0.1197**
	0.4	-0.0516	0.1968	0.2035	-0.0486	0.1255	0.1266	0.0150	0.1202	0.1211
		-0.0527	**0.1962**	**0.2031**	**-0.0507**	**0.1251**	**0.1261**	**0.0143**	**0.1198**	**0.1207**
150	0.0	-0.0228	0.1449	0.1467	-0.0228	0.0935	0.0939	0.0088	0.0935	0.0939
		-0.0228	**0.1449**	**0.1467**	**-0.0228**	**0.0935**	**0.0939**	**0.0088**	**0.0935**	**0.0939**
	0.1	-0.0262	0.1463	0.1486	-0.0258	0.0931	0.0933	0.0060	0.0921	0.0923
		-0.0265	**0.1462**	**0.1485**	**-0.0263**	**0.0930**	**0.0932**	**0.0059**	**0.0921**	**0.0922**
	0.2	-0.0281	0.1490	0.1516	-0.0264	0.0945	0.0947	0.0060	0.0926	0.0928
		-0.0286	**0.1487**	**0.1515**	**-0.0273**	**0.0944**	**0.0946**	**0.0057**	**0.0924**	**0.0926**
	0.3	-0.0292	0.1527	0.1555	-0.0271	0.0962	0.0965	0.0061	0.0937	0.0939
		-0.0300	**0.1524**	**0.1553**	**-0.0286**	**0.0961**	**0.0963**	**0.0057**	**0.0935**	**0.0937**
	0.4	-0.0305	0.1556	0.1585	-0.0280	0.0980	0.0982	0.0060	0.0945	0.0946
		-0.0316	**0.1550**	**0.1582**	**-0.0302**	**0.0066**	**0.0068**	**0.0054**	**0.0942**	**0.0944**
200	0.0	-0.0192	0.1242	0.1257	-0.0192	0.0803	0.0809	0.0090	0.0803	0.0809
		-0.0192	**0.1242**	**0.1257**	**-0.0192**	**0.0803**	**0.0809**	**0.0090**	**0.0803**	**0.0809**
	0.1	-0.0184	0.1261	0.1274	-0.0180	0.0821	0.0825	0.0084	0.0811	0.0815
		-0.0186	**0.1260**	**0.1273**	**-0.0185**	**0.0820**	**0.0824**	**0.0082**	**0.0811**	**0.0815**
	0.2	-0.0195	0.1286	0.1300	-0.0184	0.0829	0.0833	0.0084	0.0814	0.0819
		-0.0200	**0.1284**	**0.1299**	**-0.0193**	**0.0828**	**0.0832**	**0.0081**	**0.0813**	**0.0817**
	0.3	-0.0205	0.1315	0.1331	-0.0193	0.0840	0.0844	0.0080	0.0818	0.0822
		-0.0213	**0.1312**	**0.1329**	**-0.0209**	**0.0838**	**0.0842**	**0.0076**	**0.0816**	**0.0820**
	0.4	-0.0224	0.1347	0.1365	-0.0203	0.0854	0.0858	0.0080	0.0826	0.0830
		-0.0235	**0.1342**	**0.1362**	**-0.0224**	**0.0852**	**0.0856**	**0.0074**	**0.0823**	**0.0827**

Table 3: Bias, SE and $RMSE$ for $\hat{\gamma}$ based on MLE method without midpoint imputation (with midpoint imputation is in bold)

$\boldsymbol{n}$	cp	$\boldsymbol{k}$ = 12			$\boldsymbol{k}$ = 24			$\boldsymbol{k}$ = 36		
		bias	**SE**	**RMSE**	**bias**	**SE**	**bias**	**bias**	**SE**	**RMSE**
30	0.0	0.00834	0.01546	0.01757	0.00834	0.01546	0.01757	0.00834	0.01546	0.01757
		0.00834	**0.01546**	**0.01757**	**0.00834**	**0.01546**	**0.01757**	**0.00834**	**0.01546**	**0.01757**
	0.1	0.00838	0.01566	0.01776	0.00818	0.01551	0.01754	0.00791	0.01539	0.01730
		0.00838	**0.01565**	**0.01775**	**0.00820**	**0.01549**	**0.01753**	**0.00793**	**0.01537**	**0.01730**
	0.2	0.00932	0.01684	0.01925	0.00880	0.01638	0.01859	0.00808	0.01588	0.01781
		0.00934	**0.01681**	**0.01923**	**0.00884**	**0.01634**	**0.01858**	**0.00793**	**0.01537**	**0.01730**
	0.3	0.01048	0.01831	0.02110	0.00950	0.01754	0.01995	0.00816	0.01631	0.01823
		0.01051	**0.01826**	**0.02107**	**0.00956**	**0.01749**	**0.01993**	**0.00824**	**0.01626**	**0.01823**
	0.4	0.01195	0.02034	0.02359	0.01034	0.01903	0.02165	0.00826	0.01691	0.01882
		0.01199	**0.02026**	**0.02354**	**0.01043**	**0.01896**	**0.02164**	**0.00837**	**0.01684**	**0.01881**

Table 3 (*Continued*)

50	0.0	0.0043	0.01060	0.01144	0.00430	0.01060	0.01144	0.00430	0.0106	0.01144
		0.0043	**0.01060**	**0.01144**	**0.00430**	**0.01060**	**0.01144**	**0.00430**	**0.0106**	**0.01144**
	0.1	0.00483	0.01143	0.01241	0.00473	0.01139	0.01234	0.00458	0.01125	0.01215
		0.00484	**0.01143**	**0.01241**	**0.00475**	**0.01139**	**0.01234**	**0.00461**	**0.01125**	**0.01215**
	0.2	0.00532	0.01211	0.01323	0.00512	0.01200	0.01305	0.00474	0.01163	0.01256
		0.00535	**0.01210**	**0.01323**	**0.00516**	**0.01199**	**0.01305**	**0.00479**	**0.01161**	**0.01256**
	0.3	0.00602	0.01293	0.01427	0.00565	0.01267	0.01388	0.00502	0.01208	0.01308
		0.00606	**0.0129**	**0.01426**	**0.00572**	**0.01265**	**0.01388**	**0.0051**	**0.01205**	**0.01309**
	0.4	0.00691	0.01406	0.01566	0.00616	0.01344	0.01478	0.00516	0.01249	0.01351
		0.00695	**0.01401**	**0.01564**	**0.00626**	**0.0134**	**0.01479**	**0.00528**	**0.01245**	**0.01353**
100	0.0	0.00231	0.00744	0.0078	0.00231	0.00744	0.0078	0.00231	0.00744	0.0078
		0.00231	**0.00744**	**0.0078**	**0.00231**	**0.00744**	**0.0078**	**0.00231**	**0.00744**	**0.0078**
	0.1	0.00277	0.00754	0.00804	0.00274	0.00752	0.008	0.00268	0.00748	0.00795
		0.00278	**0.00754**	**0.00804**	**0.00276**	**0.00751**	**0.008**	**0.00270**	**0.00748**	**0.00795**
	0.2	0.00301	0.00786	0.00842	0.00293	0.0078	0.00834	0.00278	0.00771	0.0082
		0.00304	**0.00785**	**0.00842**	**0.00297**	**0.00779**	**0.00834**	**0.00283**	**0.00771**	**0.00821**
	0.3	0.00323	0.00831	0.00892	0.00307	0.00818	0.00874	0.00280	0.00798	0.00846
		0.00327	**0.00829**	**0.00892**	**0.00314**	**0.00816**	**0.00875**	**0.00288**	**0.00797**	**0.00847**
	0.4	0.00364	0.0088	0.00952	0.00336	0.00862	0.00925	0.00290	0.0083	0.00879
		0.00369	**0.00877**	**0.00952**	**0.00346**	**0.0086**	**0.00927**	**0.00302**	**0.00828**	**0.00881**
150	0.0	0.00153	0.00586	0.00606	0.00153	0.00586	0.00606	0.00153	0.00586	0.00606
		0.00153	**0.00586**	**0.00606**	**0.00153**	**0.00586**	**0.00606**	**0.00153**	**0.00586**	**0.00606**
	0.1	0.00158	0.00607	0.00627	0.00157	0.00604	0.00624	0.00152	0.00598	0.00617
		0.00159	**0.00607**	**0.00627**	**0.00159**	**0.00603**	**0.00624**	**0.00154**	**0.00598**	**0.00617**
	0.2	0.0017	0.00634	0.00657	0.00164	0.00629	0.00651	0.00156	0.00619	0.00638
		0.00172	**0.00633**	**0.00656**	**0.00169**	**0.00629**	**0.00651**	**0.0016**	**0.00619**	**0.00639**
	0.3	0.00183	0.00666	0.00691	0.00174	0.00658	0.00681	0.00158	0.00642	0.00661
		0.00186	**0.00665**	**0.0069**	**0.00181**	**0.00657**	**0.00681**	**0.00166**	**0.00641**	**0.00662**
	0.4	0.00199	0.00699	0.00727	0.00185	0.00689	0.00713	0.00158	0.00661	0.0068
		0.00204	**0.00698**	**0.00727**	**0.00195**	**0.00687**	**0.00714**	**0.00169**	**0.0066**	**0.00681**
200	0.0	0.00116	0.00501	0.00514	0.00116	0.00501	0.00514	0.00116	0.00501	0.00514
		0.00116	**0.00501**	**0.00514**	**0.00116**	**0.00501**	**0.00514**	**0.00116**	**0.00501**	**0.00514**
	0.1	0.00127	0.00509	0.00524	0.00124	0.00509	0.00523	0.00121	0.00506	0.0052
		0.00128	**0.00508**	**0.00524**	**0.00126**	**0.00508**	**0.00524**	**0.00124**	**0.00506**	**0.00521**
	0.2	0.00134	0.00524	0.00541	0.00129	0.00523	0.00539	0.00122	0.00518	0.00532
		0.00137	**0.00523**	**0.00541**	**0.00133**	**0.00523**	**0.00539**	**0.00127**	**0.00517**	**0.00532**
	0.3	0.00149	0.0055	0.0057	0.00141	0.00548	0.00566	0.00127	0.00538	0.00553
		0.00153	**0.00549**	**0.0057**	**0.00148**	**0.00547**	**0.00567**	**0.00134**	**0.00537**	**0.00554**
	0.4	0.00166	0.00583	0.00606	0.00153	0.00578	0.00598	0.00129	0.00563	0.00578
		0.00172	**0.00581**	**0.00606**	**0.00163**	**0.00577**	**0.00599**	**0.0014**	**0.00562**	**0.00579**

4. Hypothesis Testing and Power Analysis

Hypothesis testing is widely employed by statisticians as a formal procedure to draw conclusions about a population parameter. In this study, we are interested in testing the significance of the parameter, β_1 and parameter, γ. Lee and Wang (2003) stated the hazard of the Gompertz distribution reduces to a constant when γ=0, in equation (3), which implies that the exponential distribution will be more appropriate for the data. To test the hypotheses above, the Wald and Likelihood ratio test were used. Here, the hypothesis that we interested in testing for the parameter β_1 is,

$H_0: \beta_1 = 0$ (Covariate has no significant effect)

$H_1: \beta_1 \neq 0$ (Covariate has a significant effect)

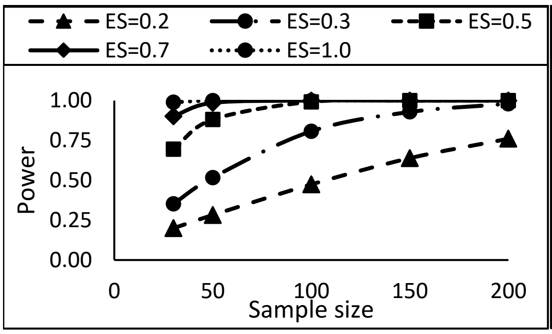
and for the parameter γ is,

$H_0: \gamma = 0$ (Exponential distribution fits the data well)
$H_1: \gamma > 0$ (Gompertz distribution fits the data well).

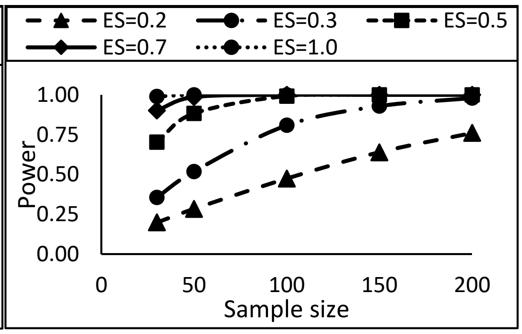
Power analysis was used in this study to evaluate the best hypothesis test for the parameters of the Gompertz model with a covariate for right and interval censored data. In power analysis, the data is simulated under alternative hypothesis by choosing the parameter β_1 and γ to be any value other than zero. The Wald and LR statistics were computed for each replication and the empirical power was obtained by taking the number of times that the test rejects the null hypothesis, divided by the number of replications, N = 5000. The empirical power for the Wald and LR test was calculated based on the proportion of rejections of the null hypothesis at two significant levels values, which were α = 0.05 and α = 0.10.

Effect size is the difference between hypothesized value H_0 and true parameter value under H_1. The greater the difference between the true value of a parameter and the value proposed in the H_0, the greater the power of the test. Five effect values were specified under H_0 for parameter β_1 and γ which were (0.2, 0.3, 0.5, 0.7, 1.0) and (0.01, 0.03, 0.05, 0.1, 0.2) respectively. The sample size, n = 30, 50, 100, 150 and 200, approximate censoring proportion, cp = 0.1, 0.2 and 0.4 and study periods k = 12, 24 and 36 months were employed in this study. Tables 4 to 15 shows the empirical power results for parameter β_1 and γ based on the MLE method with midpoint imputation to compare the performances of the Wald and LR test.

Figure 1 clearly shows that the empirical power for both the tests for the parameter β_1 increase with the increase in sample size and effect size, for α = 0.05, *cp*=0.1 and *k*= 12. The same trend was observed across other levels of *cp*. Figure 2 shows that the empirical power for both the tests were higher at lower censoring proportions for α = 0.05, effect size=0.2 and *k*= 12. Similar patterns were noted across other effect sizes. Figure 3 shows that the empirical power for both the tests were higher at higher study periods, *k* for α = 0.05, *cp*=0.4 and effect size=0.2. The same trend was observed for other effect sizes, see Table 4 (α = 0.05) and Table 5 (α = 0.10) more detailed results.

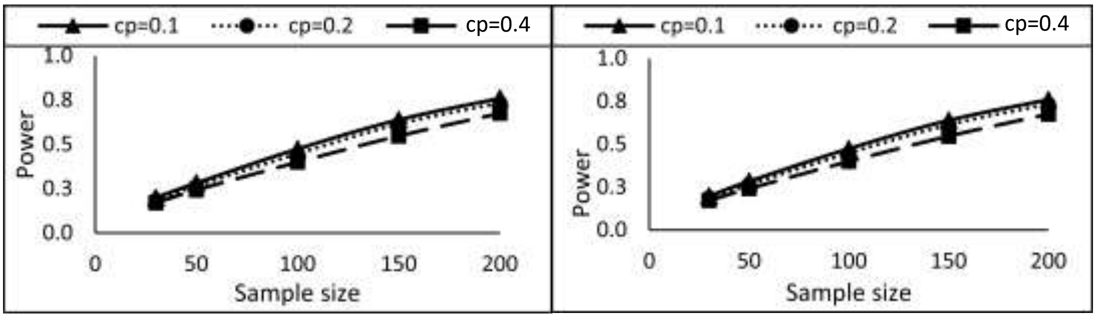


(a)



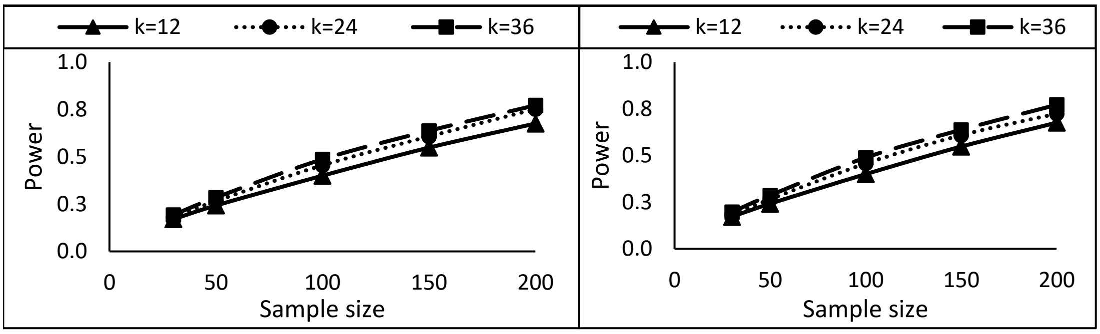
(b)

Figure 1: Empirical power of Wald (a) and LR (b) test for parameter β_1 at various effect sizes, for α = 0.05, *cp*=0.1 and k = 12



(a) (b)

Figure 2: Empirical power of Wald (a) and LR (b) test for parameter β_1 at various censoring proportions, for α = 0.05, effect size=0.2 and k = 12



(a) (b)

Figure 3: Empirical power of Wald (a) and LR (b) test for parameter β_1 at various study periods, for α = 0.05, *cp*=0.4 and effect size=0.2

More detailed result of the simulation study is given in Tables 4-15. Tables 4-9 display the empirical power for testing β_1, at different nominal levels, effect sizes, and censoring proportions. It can be observed that the empirical power clearly increases as the sample sizes and effect sizes increase but decrease as the censoring proportion increases. The empirical power was also higher at α = 0.10 compared to α = 0.05, at all sample sizes, censoring proportions, and effect sizes. The LR demonstrates greater empirical power compared to the Wald test, for both parameters, especially when effect size=0.1 and n<100. Their performances start to becomes very close when effect size>0.3, and n>100. Higher study periods seem to generate high empirical power. This was expected as higher study periods typically contain more information.

Tables 10-15 display the empirical power for testing the parameter γ, at different nominal levels, effect sizes, and censoring proportions. Similarly, the empirical power was higher at higher sample sizes and effect sizes but lower as the censoring proportion increases. The empirical power was also higher at α = 0.10 compared to α = 0.05, at all sample sizes, censoring proportions, and effect sizes. However, for the parameter γ the Wald outperforms the LR, especially when effect size=0.01, or when n<100 and effect size<0.1. Their performances start to become rather close when effect size>0.05, or when n>100 and effect size>0.01. Based on the results, we can conclude that the LR test performed better than the Wald test for parameter β_1, whereas the Wald performed better than the LR test for the parameter γ. Cohen (1988) considered 0.80 as good power for a test. In general, the empirical power for both the tests increases with the decrease in censoring proportion whereas the power increase with the increase in sample size, study period, effect size and significance level.

Table 4: Empirical power of Wald and LR test for parameter β_1 at α = 0.05 and k = 12

Test			**Wald**					**Likelihood**				
Effect Size			**0.2**	**0.3**	**0.5**	**0.7**	**1.0**	**0.2**	**0.3**	**0.5**	**0.7**	**1.0**
k=12	n=30	cp=0.1	0.1994	0.3522	0.6956	0.9020	0.9890	0.1982	0.3556	0.7024	0.9042	0.9896
		cp=0.2	0.1856	0.3314	0.6688	0.8838	0.9850	0.1882	0.3334	0.6762	0.8886	0.9868
		cp=0.4	0.1700	0.2950	0.6112	0.8456	0.9768	0.1718	0.2978	0.6166	0.8526	0.9780
	n=50	cp=0.1	0.2822	0.5176	0.8810	0.9862	0.9998	0.2840	0.5188	0.8834	0.9864	0.9998
		cp=0.2	0.2648	0.4992	0.8664	0.9810	0.9996	0.2658	0.5018	0.8666	0.9822	0.9996
		cp=0.4	0.2426	0.4324	0.8102	0.9672	0.9994	0.2414	0.4348	0.8134	0.9678	0.9994
	n=100	cp=0.1	0.4744	0.8078	0.9924	1.0000	1.0000	0.4740	0.8082	0.9926	1.0000	1.0000
		cp=0.2	0.4470	0.7784	0.9882	1.0000	1.0000	0.4480	0.7774	0.9884	1.0000	1.0000
		cp=0.4	0.3998	0.7020	0.9788	0.9998	1.0000	0.3996	0.7024	0.9792	1.0000	1.0000
	n=150	cp=0.1	0.6390	0.9286	1.0000	1.0000	1.0000	0.6394	0.9290	1.0000	1.0000	1.0000
		cp=0.2	0.6148	0.9104	1.0000	1.0000	1.0000	0.6132	0.9104	1.0000	1.0000	1.0000
		cp=0.4	0.5480	0.8564	0.9990	1.0000	1.0000	0.5472	0.8582	0.9990	1.0000	1.0000
	n=200	cp=0.1	0.7604	0.9798	1.0000	1.0000	1.0000	0.7600	0.9796	1.0000	1.0000	1.0000
		cp=0.2	0.7344	0.9708	1.0000	1.0000	1.0000	0.7342	0.9708	1.0000	1.0000	1.0000
		cp=0.4	0.6754	0.9434	1.0000	1.0000	1.0000	0.6754	0.9432	1.0000	1.0000	1.0000

Table 5: Empirical power of Wald and LR test for parameter β_1 at α = 0.10 and k = 12

Test			**Wald**					**Likelihood**				
Effect Size			**0.2**	**0.3**	**0.5**	**0.7**	**1.0**	**0.2**	**0.3**	**0.5**	**0.7**	**1.0**
k=12	n=30	cp=0.1	0.2992	0.4672	0.7964	0.9456	0.9950	0.3004	0.4698	0.7984	0.9482	0.9950
		cp=0.2	0.2846	0.4502	0.7702	0.9364	0.9938	0.2858	0.4500	0.7736	0.9382	0.9932
		cp=0.4	0.2642	0.4052	0.7258	0.9032	0.9900	0.2644	0.4080	0.7286	0.9042	0.9906
	n=50	cp=0.1	0.3960	0.6416	0.9320	0.9950	1.0000	0.3974	0.6416	0.9338	0.9952	1.0000
		cp=0.2	0.3838	0.6212	0.9164	0.9918	0.9998	0.3836	0.6212	0.9174	0.9922	1.0000
		cp=0.4	0.3482	0.5646	0.8806	0.9842	0.9998	0.3484	0.5668	0.8828	0.9846	0.9998
	n=100	cp=0.1	0.6010	0.8796	0.9980	1.0000	1.0000	0.6018	0.8798	0.9980	1.0000	1.0000
		cp=0.2	0.5724	0.8596	0.9944	1.0000	1.0000	0.5720	0.8602	0.9946	1.0000	1.0000
		cp=0.4	0.5210	0.8042	0.9898	1.0000	1.0000	0.5232	0.8040	0.9900	1.0000	1.0000
	n=150	cp=0.1	0.7460	0.9620	1.0000	1.0000	1.0000	0.7456	0.9620	1.0000	1.0000	1.0000
		cp=0.2	0.7216	0.9490	1.0000	1.0000	1.0000	0.7202	0.9488	1.0000	1.0000	1.0000
		cp=0.4	0.6654	0.9212	1.0000	1.0000	1.0000	0.6642	0.9212	1.0000	1.0000	1.0000
	n=200	cp=0.1	0.8486	0.9876	1.0000	1.0000	1.0000	0.8484	0.9876	1.0000	1.0000	1.0000
		cp=0.2	0.8218	0.9848	1.0000	1.0000	1.0000	0.8216	0.9848	1.0000	1.0000	1.0000
		cp=0.4	0.7738	0.9724	1.0000	1.0000	1.0000	0.7742	0.9726	1.0000	1.0000	1.0000

Table 6: Empirical power of Wald and LR test for parameter β_1 at α = 0.05 and k = 24

Test			Wald					Likelihood				
Effect Size			0.2	0.3	0.5	0.7	1.0	0.2	0.3	0.5	0.7	1.0
		cp=0.1	0.1970	0.3562	0.7100	0.9130	0.9916	0.2004	0.3580	0.7150	0.9170	0.9916
	n=30	cp=0.2	0.1952	0.3424	0.6934	0.9032	0.9910	0.1966	0.3444	0.6990	0.9078	0.9912
		cp=0.4	0.1816	0.3212	0.6688	0.8878	0.9868	0.1838	0.3260	0.6744	0.8920	0.9882
		cp=0.1	0.2892	0.5280	0.8892	0.9894	0.9998	0.2896	0.5308	0.8908	0.9902	0.9998
	n=50	cp=0.2	0.2796	0.5152	0.8806	0.9880	1.0000	0.2828	0.5164	0.8820	0.9878	1.0000
		cp=0.4	0.2656	0.4916	0.8618	0.9830	0.9998	0.2650	0.4932	0.8632	0.9830	1.0000
		cp=0.1	0.4924	0.8184	0.9946	1.0000	1.0000	0.4948	0.8200	0.9950	1.0000	1.0000
k=24	n=100	cp=0.2	0.4770	0.8076	0.9926	1.0000	1.0000	0.4770	0.8084	0.9934	1.0000	1.0000
		cp=0.4	0.4550	0.7716	0.9900	1.0000	1.0000	0.4564	0.7736	0.9900	1.0000	1.0000
		cp=0.1	0.6520	0.9348	1.0000	1.0000	1.0000	0.6508	0.9356	1.0000	1.0000	1.0000
	n=150	cp=0.2	0.6390	0.9272	1.0000	1.0000	1.0000	0.6392	0.9276	1.0000	1.0000	1.0000
		cp=0.4	0.6068	0.9108	1.0000	1.0000	1.0000	0.6080	0.9108	1.0000	1.0000	1.0000
		cp=0.1	0.7742	0.9808	1.0000	1.0000	1.0000	0.7750	0.9810	1.0000	1.0000	1.0000
	n=200	cp=0.2	0.7586	0.9780	1.0000	1.0000	1.0000	0.7572	0.9782	1.0000	1.0000	1.0000
		cp=0.4	0.2520	0.9678	1.0000	1.0000	1.0000	0.7238	0.9678	1.0000	1.0000	1.0000

Table 7: Empirical power of Wald and LR test for parameter β_1 at α = 0.10 and k = 24

Test			Wald					Likelihood				
Effect Size			0.2	0.3	0.5	0.7	1.0	0.2	0.3	0.5	0.7	1.0
		cp=0.1	0.2986	0.4774	0.8046	0.9534	0.9956	0.3008	0.4806	0.8066	0.9538	0.9956
	n=30	cp=0.2	0.2906	0.4674	0.7920	0.9528	0.9960	0.2920	0.4694	0.7930	0.9544	0.9964
		cp=0.4	0.2968	0.4440	0.7750	0.9392	0.9946	0.2764	0.4466	0.7774	0.9410	0.9954
		cp=0.1	0.4042	0.6504	0.9384	0.9958	1.0000	0.4038	0.6524	0.9402	0.9960	1.0000
	n=50	cp=0.2	0.3956	0.6420	0.9318	0.9948	1.0000	0.3970	0.6432	0.9324	0.9954	1.0000
		cp=0.4	0.3732	0.6138	0.9184	0.9932	1.0000	0.3756	0.6146	0.9190	0.9936	1.0000
		cp=0.1	0.6140	0.8878	0.9982	1.0000	1.0000	0.6150	0.8882	0.9982	1.0000	1.0000
k=24	n=100	cp=0.2	0.6002	0.8790	0.9974	1.0000	1.0000	0.5996	0.8800	0.9974	1.0000	1.0000
		cp=0.4	0.5786	0.8576	0.9962	1.0000	1.0000	0.5784	0.8592	0.9960	1.0000	1.0000
		cp=0.1	0.7578	0.9672	1.0000	1.0000	1.0000	0.7588	0.9672	1.0000	1.0000	1.0000
	n=150	cp=0.2	0.7482	0.9624	1.0000	1.0000	1.0000	0.7470	0.9632	1.0000	1.0000	1.0000
		cp=0.4	0.7208	0.9508	1.0000	1.0000	1.0000	0.7200	0.9510	1.0000	1.0000	1.0000
		cp=0.1	0.8630	0.9898	1.0000	1.0000	1.0000	0.8622	0.9896	1.0000	1.0000	1.0000
	n=200	cp=0.2	0.8474	0.9882	1.0000	1.0000	1.0000	0.8486	0.9884	1.0000	1.0000	1.0000
		cp=0.4	0.8200	0.9832	1.0000	1.0000	1.0000	0.8204	0.9832	1.0000	1.0000	1.0000

Table 8: Empirical power of Wald and LR test for parameter β_1 at α = 0.05 and k = 36

Test			**Wald**					**Likelihood**				
Effect Size			**0.2**	**0.3**	**0.5**	**0.7**	**1.0**	**0.2**	**0.3**	**0.5**	**0.7**	**1.0**
		cp=0.1	0.1996	0.3602	0.7114	0.9174	0.9918	0.2044	0.3636	0.7198	0.9210	0.9926
	n=30	*cp*=0.2	0.1992	0.3550	0.7066	0.9152	0.9910	0.2036	0.3596	0.7130	0.9196	0.9916
		cp=0.4	0.1930	0.3466	0.7002	0.9064	0.9912	0.1970	0.3500	0.7014	0.9092	0.9914
		cp=0.1	0.2944	0.5322	0.8976	0.9900	1.0000	0.2954	0.5338	0.8992	0.9906	1.0000
	n=50	*cp*=0.2	0.2920	0.5284	0.8942	0.9892	1.0000	0.2946	0.5298	0.8964	0.9900	1.0000
		cp=0.4	0.2846	0.5148	0.8886	0.9880	1.0000	0.2860	0.5150	0.8894	0.9892	1.0000
		cp=0.1	0.5036	0.8302	0.9944	1.0000	1.0000	0.5040	0.8302	0.9948	1.0000	1.0000
k=36	*n*=100	*cp*=0.2	0.4970	0.8258	0.9932	1.0000	1.0000	0.4974	0.8248	0.9936	1.0000	1.0000
		cp=0.4	0.4864	0.8126	0.9930	1.0000	1.0000	0.4880	0.8132	0.9932	1.0000	1.0000
		cp=0.1	0.6568	0.9386	1.0000	1.0000	1.0000	0.6558	0.9398	1.0000	1.0000	1.0000
	n=150	*cp*=0.2	0.6524	0.9370	1.0000	1.0000	1.0000	0.6536	0.9370	1.0000	1.0000	1.0000
		cp=0.4	0.6370	0.9322	1.0000	1.0000	1.0000	0.6370	0.9324	1.0000	1.0000	1.0000
		cp=0.1	0.7850	0.9826	1.0000	1.0000	1.0000	0.7856	0.9824	1.0000	1.0000	1.0000
	n=200	*cp*=0.2	0.7784	0.9804	1.0000	1.0000	1.0000	0.7788	0.9802	1.0000	1.0000	1.0000
		cp=0.4	0.7714	0.9772	1.0000	1.0000	1.0000	0.7710	0.9778	1.0000	1.0000	1.0000

Table 9**:** Empirical power of Wald and LR test for parameter β_1 at α = 0.10 and k = 36

Test			**Wald**					**Likelihood**				
Effect Size			**0.2**	**0.3**	**0.5**	**0.7**	**1.0**	**0.2**	**0.3**	**0.5**	**0.7**	**1.0**
		cp=0.1	0.2996	0.4852	0.8072	0.9556	0.9964	0.3008	0.4864	0.8096	0.9556	0.9970
	n=30	*cp*=0.2	0.3002	0.4830	0.8050	0.9548	0.9960	0.3008	0.4836	0.8078	0.9556	0.9968
		cp=0.4	0.2926	0.4744	0.7994	0.9518	0.9954	0.2954	0.4784	0.8008	0.9536	0.9958
		cp=0.1	0.4124	0.6590	0.9434	0.9960	1.0000	0.4122	0.6584	0.9440	0.9962	1.0000
	n=50	*cp*=0.2	0.4114	0.6540	0.9386	0.9960	1.0000	0.4100	0.6542	0.9390	0.9960	1.0000
		cp=0.4	0.3996	0.6436	0.9362	0.9950	1.0000	0.4002	0.6444	0.9374	0.9952	1.0000
		cp=0.1	0.6224	0.8924	0.9978	1.0000	1.0000	0.6214	0.8924	0.9980	1.0000	1.0000
k=36	*n*=100	*cp*=0.2	0.6156	0.8866	0.9978	1.0000	1.0000	0.6158	0.8866	0.9976	1.0000	1.0000
		cp=0.4	0.6080	0.8796	0.9974	1.0000	1.0000	0.6096	0.8794	0.9972	1.0000	1.0000
		cp=0.1	0.7662	0.9704	1.0000	1.0000	1.0000	0.7662	0.9708	1.0000	1.0000	1.0000
	n=150	*cp*=0.2	0.7634	0.9678	1.0000	1.0000	1.0000	0.7642	0.9678	1.0000	1.0000	1.0000
		cp=0.4	0.7526	0.9636	1.0000	1.0000	1.0000	0.7536	0.9640	1.0000	1.0000	1.0000
		cp=0.1	0.8646	0.9918	1.0000	1.0000	1.0000	0.8652	0.9918	1.0000	1.0000	1.0000
	n=200	*cp*=0.2	0.8646	0.9908	1.0000	1.0000	1.0000	0.8640	0.9912	1.0000	1.0000	1.0000
		cp=0.4	0.8526	0.9890	1.0000	1.0000	1.0000	0.8526	0.9890	1.0000	1.0000	1.0000

Table 10: Empirical power of Wald and LR test for parameter γ at α = 0.05 and k = 12

Test			Wald					Likelihood				
Effect Size			**0.01**	**0.03**	**0.05**	**0.10**	**0.20**	**0.01**	**0.03**	**0.05**	**0.10**	**0.20**
k=12	n=30	cp=0.1	0.5056	0.8656	0.9626	0.9970	0.9996	0.3314	0.7414	0.9076	0.9902	0.9992
		cp=0.2	0.4902	0.8504	0.9522	0.9958	1.0000	0.3180	0.7110	0.8878	0.9838	0.9996
		cp=0.4	0.4458	0.8020	0.9234	0.9888	0.9988	0.2892	0.6356	0.8238	0.9640	0.9982
	n=50	cp=0.1	0.6244	0.9682	0.9972	1.0000	1.0000	0.4560	0.9144	0.9884	1.0000	1.0000
		cp=0.2	0.6032	0.9586	0.9954	1.0000	1.0000	0.4302	0.8934	0.9824	1.0000	1.0000
		cp=0.4	0.5548	0.9290	0.9890	1.0000	1.0000	0.3736	0.8342	0.9628	1.0000	1.0000
	n=100	cp=0.1	0.8420	0.9998	1.0000	1.0000	1.0000	0.7084	0.9984	1.0000	1.0000	1.0000
		cp=0.2	0.8210	0.9994	1.0000	1.0000	1.0000	0.6736	0.9964	1.0000	1.0000	1.0000
		cp=0.4	0.7576	0.9974	1.0000	1.0000	1.0000	0.5976	0.9882	0.9998	1.0000	1.0000
	n=150	cp=0.1	0.9344	1.0000	1.0000	1.0000	1.0000	0.8614	1.0000	1.0000	1.0000	1.0000
		cp=0.2	0.9174	1.0000	1.0000	1.0000	1.0000	0.8254	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.8686	0.9998	1.0000	1.0000	1.0000	0.7398	0.9994	1.0000	1.0000	1.0000
	n=200	cp=0.1	0.9794	1.0000	1.0000	1.0000	1.0000	0.9436	1.0000	1.0000	1.0000	1.0000
		cp=0.2	0.9728	1.0000	1.0000	1.0000	1.0000	0.9282	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9398	1.0000	1.0000	1.0000	1.0000	0.8662	0.9998	1.0000	1.0000	1.0000

Table 11: Empirical power of Wald and LR test for parameter γ at α = 0.10 and k = 12

Test			Wald					Likelihood				
Effect Size			**0.01**	**0.03**	**0.05**	**0.10**	**0.20**	**0.01**	**0.03**	**0.05**	**0.10**	**0.20**
k=12	n=30	cp=0.1	0.6544	0.9366	0.9882	0.9994	1.0000	0.4786	0.8510	0.9556	0.9966	0.9996
		cp=0.2	0.6442	0.9286	0.9840	0.9988	1.0000	0.4622	0.8286	0.9438	0.9946	0.9998
		cp=0.4	0.6094	0.8936	0.9716	0.9966	0.9998	0.4148	0.7738	0.9056	0.9856	0.9988
	n=50	cp=0.1	0.7668	0.9884	0.9990	1.0000	1.0000	0.6000	0.9636	0.9966	1.0000	1.0000
		cp=0.2	0.7472	0.9838	0.9990	1.0000	1.0000	0.5754	0.9522	0.9938	1.0000	1.0000
		cp=0.4	0.7080	0.9726	0.9980	1.0000	1.0000	0.5196	0.9160	0.9868	1.0000	1.0000
	n=100	cp=0.1	0.9206	1.0000	1.0000	1.0000	1.0000	0.8262	0.9996	0.9990	1.0000	1.0000
		cp=0.2	0.9058	1.0000	1.0000	1.0000	1.0000	0.7998	0.9990	1.0000	1.0000	1.0000
		cp=0.4	0.8648	0.9988	1.0000	1.0000	1.0000	0.7292	0.9966	1.0000	1.0000	1.0000
	n=150	cp=0.1	0.9738	1.0000	1.0000	1.0000	1.0000	0.9268	1.0000	1.0000	1.0000	1.0000
		cp=0.2	0.9648	1.0000	1.0000	1.0000	1.0000	0.9084	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9390	1.0000	1.0000	1.0000	1.0000	0.8542	0.9998	1.0000	1.0000	1.0000
	n=200	cp=0.1	0.9924	1.0000	1.0000	1.0000	1.0000	0.9766	1.0000	1.0000	1.0000	1.0000
		cp=0.2	0.9882	1.0000	1.0000	1.0000	1.0000	0.9692	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9760	1.0000	1.0000	1.0000	1.0000	0.9322	1.0000	1.0000	1.0000	1.0000

Table 12: Empirical power of Wald and LR test for parameter γ at α = 0.05 and k = 24

Test			**Wald**					**Likelihood**				
Effect Size			**0.01**	**0.03**	**0.05**	**0.10**	**0.20**	**0.01**	**0.03**	**0.05**	**0.10**	**0.20**
		cp=0.1	0.5032	0.8646	0.9612	0.9972	1.0000	0.3312	0.7428	0.9060	0.9910	0.9998
	n=30	*cp*=0.2	0.4892	0.8420	0.9500	0.9968	1.0000	0.3128	0.7090	0.8862	0.9872	1.0000
		cp=0.4	0.4428	0.7878	0.9186	0.9932	1.0000	0.2722	0.6314	0.8268	0.9798	0.9998
		cp=0.1	0.6242	0.9690	0.9974	1.0000	1.0000	0.4586	0.9154	0.9880	1.0000	1.0000
	n=50	*cp*=0.2	0.6062	0.9574	0.9950	1.0000	1.0000	0.4284	0.8912	0.9808	1.0000	1.0000
		cp=0.4	0.5562	0.9302	0.9880	1.0000	1.0000	0.3648	0.8350	0.9632	0.9994	1.0000
		cp=0.1	0.8434	0.9998	1.0000	1.0000	1.0000	0.7110	0.9978	1.0000	1.0000	1.0000
k=24	*n*=100	*cp*=0.2	0.8258	0.9992	1.0000	1.0000	1.0000	0.6792	0.9960	1.0000	1.0000	1.0000
		cp=0.4	0.7606	0.9972	1.0000	1.0000	1.0000	0.5984	0.9882	0.9998	1.0000	1.0000
		cp=0.1	0.9366	1.0000	1.0000	1.0000	1.0000	0.8646	1.0000	1.0000	1.0000	1.0000
	n=150	*cp*=0.2	0.9200	1.0000	1.0000	1.0000	1.0000	0.8318	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.8748	1.0000	1.0000	1.0000	1.0000	0.7512	0.9996	1.0000	1.0000	1.0000
		cp=0.1	0.9802	1.0000	1.0000	1.0000	1.0000	0.9448	1.0000	1.0000	1.0000	1.0000
	n=200	*cp*=0.2	0.9728	1.0000	1.0000	1.0000	1.0000	0.9286	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9446	1.0000	1.0000	1.0000	1.0000	0.8752	1.0000	1.0000	1.0000	1.0000

Table 13: Empirical power of Wald and LR test for parameter γ at α = 0.10 and k = 24

Test			**Wald**					**Likelihood**				
Effect Size			**0.01**	**0.03**	**0.05**	**0.10**	**0.20**	**0.01**	**0.03**	**0.05**	**0.10**	**0.20**
		cp=0.1	0.6548	0.9360	0.9880	0.9996	1.0000	0.4752	0.8490	0.9612	0.9968	1.0000
	n=30	*cp*=0.2	0.6434	0.9254	0.9822	0.9994	1.0000	0.4606	0.8272	0.9420	0.9962	1.0000
		cp=0.4	0.6034	0.8896	0.9636	0.9984	1.0000	0.4054	0.7630	0.9072	0.9922	1.0000
		cp=0.1	0.7678	0.9884	0.9992	1.0000	1.0000	0.5978	0.9618	0.9964	1.0000	1.0000
	n=50	*cp*=0.2	0.7466	0.9844	0.9992	1.0000	1.0000	0.5768	0.9512	0.9936	1.0000	1.0000
		cp=0.4	0.7068	0.9714	0.9960	1.0000	1.0000	0.5192	0.9164	0.9858	1.0000	1.0000
		cp=0.1	0.9218	1.0000	1.0000	1.0000	1.0000	0.8276	0.9994	1.0000	1.0000	1.0000
k=24	*n*=100	*cp*=0.2	0.9078	1.0000	1.0000	1.0000	1.0000	0.8072	0.9990	1.0000	1.0000	1.0000
		cp=0.4	0.8704	0.9990	1.0000	1.0000	1.0000	0.7604	0.9966	1.0000	1.0000	1.0000
		cp=0.1	0.9754	1.0000	1.0000	1.0000	1.0000	0.9292	1.0000	1.0000	1.0000	1.0000
	n=150	*cp*=0.2	0.9672	1.0000	1.0000	1.0000	1.0000	0.9120	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9424	1.0000	1.0000	1.0000	1.0000	0.8586	1.0000	1.0000	1.0000	1.0000
		cp=0.1	0.9934	1.0000	1.0000	1.0000	1.0000	0.9772	1.0000	1.0000	1.0000	1.0000
	n=200	*cp*=0.2	0.9904	1.0000	1.0000	1.0000	1.0000	0.9688	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9776	1.0000	1.0000	1.0000	1.0000	0.9364	1.0000	1.0000	1.0000	1.0000

Table 14**:** Empirical power of Wald and LR test for parameter γ at α = 0.05 and k = 36

Test			**Wald**					**Likelihood**				
Effect Size			**0.01**	**0.03**	**0.05**	**0.10**	**0.20**	**0.01**	**0.03**	**0.05**	**0.10**	**0.20**
		cp=0.1	0.4982	0.8636	0.9610	0.9982	1.0000	0.3340	0.7428	0.9096	0.9930	0.9998
	n=30	*cp*=0.2	0.4824	0.8440	0.9506	0.9978	1.0000	0.3128	0.7158	0.8944	0.9934	1.0000
		cp=0.4	0.4312	0.7908	0.9358	0.9978	1.0000	0.2702	0.6510	0.8640	0.9930	0.9998
		cp=0.1	0.6246	0.9678	0.9974	1.0000	1.0000	0.4578	0.9166	0.9892	1.0000	1.0000
	n=50	*cp*=0.2	0.6026	0.9582	0.9952	1.0000	1.0000	0.4276	0.8976	0.9840	1.0000	1.0000
		cp=0.4	0.5508	0.9290	0.9918	1.0000	1.0000	0.3606	0.8460	0.9754	1.0000	1.0000
		cp=0.1	0.8446	0.9998	1.0000	1.0000	1.0000	0.7126	0.9974	1.0000	1.0000	1.0000
k=36	*n*=100	*cp*=0.2	0.8218	0.9996	1.0000	1.0000	1.0000	0.6816	0.9950	1.0000	1.0000	1.0000
		cp=0.4	0.7618	0.9962	1.0000	1.0000	1.0000	0.5972	0.9892	1.0000	1.0000	1.0000
		cp=0.1	0.9386	1.0000	1.0000	1.0000	1.0000	0.8630	1.0000	1.0000	1.0000	1.0000
	n=150	*cp*=0.2	0.9244	1.0000	1.0000	1.0000	1.0000	0.8320	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.8742	1.0000	1.0000	1.0000	1.0000	0.7566	0.9992	1.0000	1.0000	1.0000
		cp=0.1	0.9800	1.0000	1.0000	1.0000	1.0000	0.9452	1.0000	1.0000	1.0000	1.0000
	n=200	*cp*=0.2	0.9734	1.0000	1.0000	1.0000	1.0000	0.9290	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9450	1.0000	1.0000	1.0000	1.0000	0.8782	1.0000	1.0000	1.0000	1.0000

Table 15: Empirical power of Wald and LR test for parameter γ at α = 0.10 and k = 36

Test			**Wald**					**Likelihood**				
Effect Size			**0.01**	**0.03**	**0.05**	**0.10**	**0.20**	**0.01**	**0.03**	**0.05**	**0.10**	**0.20**
		cp=0.1	0.6514	0.9346	0.9886	0.9998	1.0000	0.4982	0.8472	0.9566	0.9976	1.0000
	n=30	*cp*=0.2	0.6366	0.9206	0.9836	0.9998	1.0000	0.4564	0.8282	0.9464	0.9976	1.0000
		cp=0.4	0.5908	0.8896	0.9734	0.9998	1.0000	0.3990	0.7734	0.9282	0.9974	1.0000
		cp=0.1	0.7684	0.9876	0.9994	1.0000	1.0000	0.5968	0.9622	0.9970	1.0000	1.0000
	n=50	*cp*=0.2	0.7452	0.9830	0.9992	1.0000	1.0000	0.5738	0.9514	0.9948	1.0000	1.0000
		cp=0.4	0.6994	0.9732	0.9978	1.0000	1.0000	0.5168	0.9194	0.9904	1.0000	1.0000
		cp=0.1	0.9218	1.0000	1.0000	1.0000	1.0000	0.8278	0.9998	0.9990	1.0000	1.0000
k=36	*n*=100	*cp*=0.2	0.9070	1.0000	1.0000	1.0000	1.0000	0.8048	0.9990	1.0000	1.0000	1.0000
		cp=0.4	0.8666	0.9990	1.0000	1.0000	1.0000	0.7356	0.9952	1.0000	1.0000	1.0000
		cp=0.1	0.9758	1.0000	1.0000	1.0000	1.0000	0.9294	1.0000	1.0000	1.0000	1.0000
	n=150	*cp*=0.2	0.9674	1.0000	1.0000	1.0000	1.0000	0.9132	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9432	1.0000	1.0000	1.0000	1.0000	0.8598	1.0000	1.0000	1.0000	1.0000
		cp=0.1	0.9936	1.0000	1.0000	1.0000	1.0000	0.9770	1.0000	1.0000	1.0000	1.0000
	n=200	*cp*=0.2	0.9902	1.0000	1.0000	1.0000	1.0000	0.9692	1.0000	1.0000	1.0000	1.0000
		cp=0.4	0.9796	1.0000	1.0000	1.0000	1.0000	0.9380	1.0000	1.0000	1.0000	1.0000

5. Real Data Analysis

In this section, the breast cosmetic real data described in Finkelstein (1986) was explored and analyzed, with the treatment as the covariate. 46 subjects were treated with radiation alone (Treatment=1) and 48 were treated with radiation plus chemotherapy (Treatment=0). Among 94 patients, 56 lifetimes were interval-censored and 38 were right-censored. Figure 4 displays the plot of the estimated Kaplan-Meier survival probabilities overlapped with the survival probabilities obtained using the Gompertz model. The plot indicates that the Gompertz distribution may be appropriate for the breast cancer patient data.

Table 16 and Figure 5 suggests that there was a difference in time to first moderate or severe breast retraction between patients under Treatment 0 and Treatment 1. However, to check this assumption with the formal statistical test, Wald and LR tests were conducted for the covariate parameter, β_1. In Table 18, the results of the Wald and LR test statistics show a significant treatment effect at α = 0.10 and α = 0.05. Hence, we can conclude that there is a significant treatment effect at α = 0.05. Following that, for parameter, γ, the null hypothesis was rejected at α = 0.05 and α = 0.10 for both the Wald and LR tests. From the result in Table 17, we can see that the confidence intervals for parameter β_1 and γ agree with the test results.

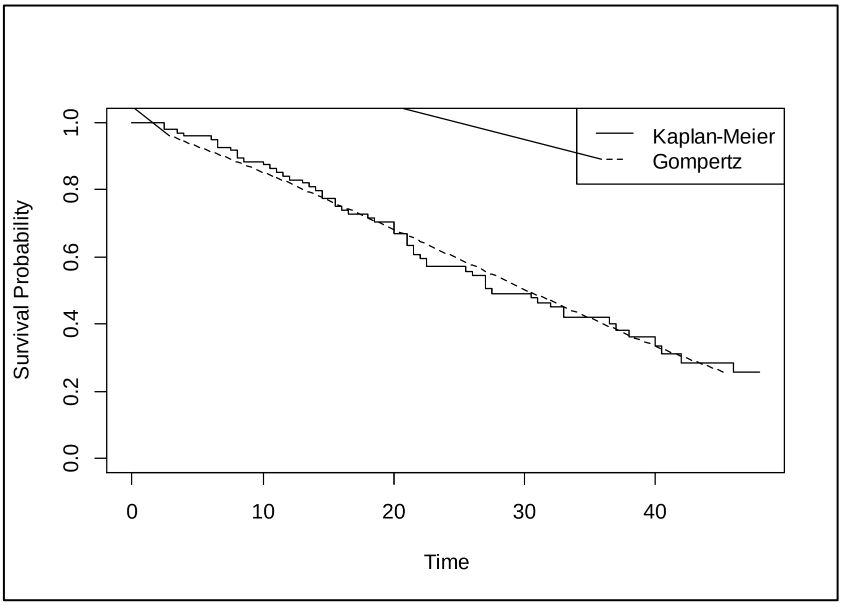


Figure 4: Plot of survivor functions for breast cancer patients data

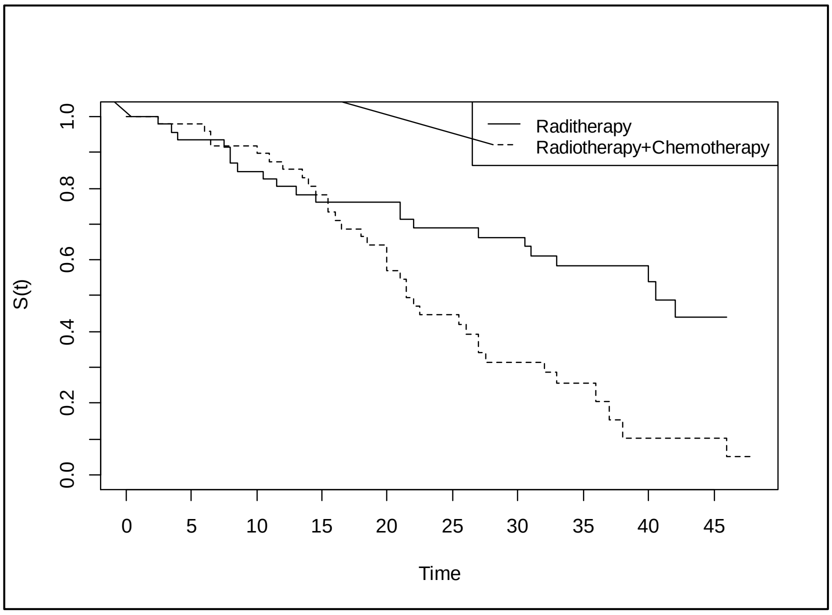


Figure 5**:** Kaplan-Meier plot for breast cancer patients by treatment.

Table 16: Descriptive statistics for breast cancer patients by treatment

Treatment	***n***	**Events**	**Median**	**0.95 LCL**	**0.95 UCL**
1	46	21	39.0	33.5	NA
0	48	35	19.5	18.5	30.5

Table 17: Parameter estimates of the Gompertz Distribution using MLE (midpoint imputation)

Parameter	Coefficient	Std. Error	90% Wald CI	95% Wald CI
β_1	0.9324	0.28390	(0.4653, 1.3993)	(0.3759, 1.4888)
γ	0.0399	0.01139	(0.0211, 0.0586)	(0.0175, 0.0622)

Table 18: Hypothesis testing based on Wald and LR test

Parameter	Wald	LR	$\alpha = 0.10$	$\alpha = 0.05$
$H_0: \beta_1 = 0$ $H_1: \beta_1 \neq 0$	3.284	11.246	Both test significant	Both tests significant
$H_0: \gamma = 0$ $H_1: \gamma > 0$	3.503	11.652	Both test significant	Both test significant

6. Conclusion and Future work

In this research, the Gompertz distribution was extended to incorporate a fixed covariate with right and interval censored data. The evaluation was carried out by conducting a simulation study to compare the performance MLE methods (with and without midpoint imputation) for this model at the various censoring proportion (*cp*), sample sizes (*n*) and study periods (*k*) by computing the values of bias, SE and RMSE. Based on the stimulation study, the MLE method with midpoint imputation gave better performance in estimating the parameters of Gompertz distribution.

The performances of the Wald and LR tests were compared for the parameters of this model via power analysis at different censoring proportions, sample sizes, study periods, significant levels and effect sizes. In general, the statistical power for testing the parameters β_1 and γ increased for both tests when there was a decrement in censoring proportions and increment in sample sizes, study periods, significant levels, and effect sizes. The LR test performed better for parameter β_1 as its empirical power was slightly higher as compared to the Wald test especially when $n<100$ and effect size was low. The Wald performed better than the LR for testing of parameter γ, also when $n<100$ and effect size was low. Both tests tend to perform equally well for larger samples and effect sizes.

In real data analysis, the results showed that Gompertz distribution fitted the breast cancer data well, which was supported by the result of the hypothesis test for parameters γ using both Wald and LR tests. The hypothesis testing for parameters β_1 revealed a statistically significant treatment effect at $\alpha = 0.05$, as determined by both Wald and LR tests. The current work only focused on comparing the performance of midpoint imputation method. Hence, several other popular imputation techniques such as right point imputation, left point imputation and random point imputation can be employed and compared in the future. The model only focused on the right- and interval-censored data. The current research can also be extended to include left censored and truncated data. In addition, more covariates can be included in the model and the model can easily be extended to include time-varying covariates.

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