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Fractional growth model of abalone length

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ABSTRACT

This paper uses fractional growth model modified from McKendrick equation to describe the growth of abalone length. The fractional model is analyzed by generalized differential transform method to obtain Taylor's series which is then used to predict the abalone length growth. The results are indicated by fractional order equal to 0.8. The results also show that by simulating the series with fractional order and integer order, the fractional model provides more robust results than the model with integer order.

1. Introduction

Abalone, a type of seashell, is extensively farmed in various countries, including Indonesia. It is not only cherished as a favored food item due to its high protein content but is also crafted into attractive souvenirs using its shells. Nonetheless, abalone cultivation demands a significant amount of time because of its slow growth. Moreover, the length of the abalone plays a crucial role in determining its price. Consequently, a growth model for abalone length was presented in the form of an ordinary differential equation as follows.[1]

$$l(t) = 12\left(1 - e^{-0.04305(t+0.0537509)}\right), \tag{1}$$

where $l(t)$ represents the length of abalone at time t. However, the growth model has been expanded in different ways including the McKendrick equations. The McKendrick equation stands out as a widely recognized growth model for partial differential equations. Originally presented by McKendrick, this model emerged from his resolution of an ecological challenge related to a singular population with age structure.[2] Let $w(s,t)$ be a density of individuals of age s at time t, then

$$\frac{\partial w(s,t)}{\partial t} + \frac{\partial w(s,t)}{\partial s} = -\mu(s,t)w(s,t), \tag{2}$$

with $\mu(s,t)$ denotes the mortality rate which is dependent of time. The Eq. (2) is well known as McKendrick model. The model provides more detailed prediction of population growth. For instance, it predicted the Swedish female population (1831–1925) properly, showing age patterns for decreasing birth rates and increasing deaths.[3] In addition, the model indicate that the growth rate of a population at immature and adults stages is different due to consumption ability and limited resources.[4] Also, many cases of disease or virus spread have been modeled based on age-structure to obtain more accurate results.[5–8] Furthermore, the model has been also modified as formulation of hiring age and retirement age of workers as well as modified into goodwill model for length of product life cycle.[9,10]

On the other side, fractional integral and derivative have played an essential role in expansion of mathematical modeling theory. Some researches indicated that mathematical modeling with fractional order more realistic than integer order. For example, the fractional partial differential equation (FPDE) of behavior the oil pollution in water provide solution more properly than the model with classical integer order.[11] The system fractional PDE of anomalous diffusion process also shows more realistic results than the system equation with integer order.[12] In addition, fractional PDE of the complex fluid flow process in confined nano-scale shale formation sufficiently to optimize shale gas recovery has presented better understanding than the PDE with integer order[13] and so on.

Therefore, this study propose the partial differential equation of growth model with fractional order to predict the growth of abalone length in more detail. The results of this study can be a reference to analyze the growth of abalone length in the further cultivation. It implies that the optimal harvesting time can be also predicted. In addition, the model is constructed as expansion of McKendrick equation and pure growth model as indicated by Arora (2023). The pure growth of particles was given as follows.

$$\frac{\partial v(s,t)}{\partial t} + \frac{\partial [h(s)v(s,t)]}{\partial s} = 0, \tag{3}$$

where $v(s,t)$ denotes the number density distribution of particles with size s at time t and $h(s)$ is the growth rate process. The solution of Eq. (3) was solved by the Adomian decomposition method which

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produced the exponential function.[14] Furthermore, the generalized differential transform method (GDTM) is one also popular method to solve the fractional differential equations as specially linear differential equations. Basically, this method was first introduced by Zhou who solved the linear and non-linear equations of electric circuit.[15] For more detail there are some equations have been solved properly by GDTM including: linear and non-linear Volterra integral equation[16]; fractional diffusion-wave equation[17]; the local diffusion equations[18]; conformable space–time PDE[19,20]; couple time fractional non-linear evolutions equations[21]; fractional generalized Burger–Fisher equations,[22] and analytical study of atmospheric internal waves model.[11] Thus, in this study the fractional partial differential equation of growth model is analyzed using GDTM.

The remainder sections of this paper are organized as follows. Section 2, describes the fundamental theory of fractional integral and derivative. Section 3, presents the generalized differential transform method. Section 4, modifies the growth model into fractional growth model. Section 5, examines the fractional growth model by predicting the growth of abalone length. Section 6, draws the conclusions.

2. Fractional integral and derivative

This part recalled several prominent definitions and properties concisely of the fractional integral and derivative as follows.

Definition 2.1 (*Ref.* 23)**.** Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. The Riemann–Liouville fractional integral and derivative with order α are defined respectively as follows.

$$I_a^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s-\tau)^{\alpha-1} f(\tau)d\tau \tag{4}$$

and

$$^{RL}D_s^\alpha f(s) = \begin{cases} \frac{d^m}{ds^m} \text{ if } \alpha = m \in \mathbb{N}, \\ \frac{d^m}{ds^m} \int_0^s \frac{(s-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} f(\tau)d\tau \text{ if } \alpha \in (m-1,m), m \in \mathbb{N}. \end{cases} \tag{5}$$

The definition (5) indicates that the derivative of constant function is not equal to zero so that the definition was modified into Caputo fractional derivatives.

Definition 2.2 (*Refs.* 24, 25)**.** Let a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Caputo fractional derivative of function f with order-α is defined as,

$$^{C}D_a^\alpha f(s) = \begin{cases} \frac{d^m}{ds^m} \text{ if } \alpha = m \in \mathbb{N}, \\ \int_a^s \frac{(s-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} f^{(m)}(\tau)d\tau \text{ if } \alpha \in (m-1,m), m \in \mathbb{N}. \end{cases} \tag{6}$$

Clearly, by the definition (6) derivative of constant function is zero. Here, some basic properties as below.

1. $\left(I_a^\alpha I_a^\beta f\right)(s) = \left(I_a^\beta I_a^\alpha f\right) = \left(I^{\alpha+\beta} f\right)(s)$,
2. $I_a^\alpha (s-a)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)}(s-a)^{\lambda+\alpha}$,
3. $\left(I_a^{\alpha\, C} D_a^\alpha f\right)(s) = f(s) - \sum_{k=0}^{m-1} f^{(k)}(a)\frac{(s-a)^k}{k!}$,

where $\alpha, \beta > 0$, $a \geq 0$, $\alpha \in (m-1,m)$, $m \in \mathbb{N}$, and $\lambda > -1$.[17]

Similarly, the definition of fractional partial derivative is written as below.

Definition 2.3 (*Ref.* 26)**.** The Caputo fractional partial derivatives of function f with order-α are defined as,

$$\frac{\partial^\alpha f(s,t)}{\partial s^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^s (s-\tau)^{n-\alpha-1} \frac{\partial^n f(\tau,t)}{\partial \tau^n} d\tau, \; n-1 < \alpha \leq n, \tag{7}$$

and

$$\frac{\partial^\alpha f(s,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(s,\tau)}{\partial \tau^n} ds, \; n-1 < \alpha \leq n. \tag{8}$$

So, the basic properties can be also generalized similarly.

3. Generalized differential transform method

In this part we provide some succinctly definition and theorems of generalized differential transform method. The definition and theorems are given as below.

Definition 3.1. Let $f(s,t)$ be analytical function that it can be presented as multi-Taylor series about (s_0,t_0) as follows:

$$f(s,t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} F(k,m)(s-s_0)^{k\alpha}(t-t_0)^{m\alpha}, \tag{9}$$

with

$$F(k,m) = \frac{1}{\Gamma(k\alpha+1)} \frac{1}{\Gamma(m\alpha+1)} \left(\frac{\partial^{(k+m)\alpha} w(s,t)}{\partial s^{k\alpha} \partial t^{m\alpha}} \right)_{(s_0,t_0)} \tag{10}$$

where $k, m = 0, 1, \ldots, n$ and $\alpha \in (0,1]$.

Here, $F(k,m)$ is called differential transform of the function $f(s,t)$. Furthermore, some theorems of GDTM are provided as follows.

Theorem 3.1.

$$\text{If } f(s,t) = au(s,t) \pm bv(s,t), \text{ then } F(k,m) = aU(k,m) + bV(k,m) \tag{11}$$

Proof. It is clear from Definition 3.1.

Theorem 3.2. *If*

$$f(s,t) = au(s,t)v(s,t),$$

then

$$F(k,m) = a\sum_{i=0}^{k} \sum_{j=0}^{m} U(k,m-j)V(k-i,m). \tag{12}$$

Proof. By Definition 3.1, we have

$$\begin{aligned} f(s,t) &= a\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (s-s_0)^{k\alpha}(t-t_0)^{m\alpha} F(k,m) \\ &= \left(a\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(s-s_0)^{k\alpha}}{\Gamma(k\alpha+1)} \frac{(t-t_0)^{m\alpha}}{\Gamma(m\alpha+1)} \frac{\partial^{(k+m)\alpha} u(s,t)}{\partial s^{k\alpha} \partial t^{m\alpha}} \right)_{s=s_0,t=t_0} \\ &\otimes \left(\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(s-s_0)^{k\alpha}}{\Gamma(k\alpha+1)} \frac{(t-t_0)^{m\alpha}}{\Gamma(m\alpha+1)} \frac{\partial^{(k+m)\alpha} v(s,t)}{\partial s^{k\alpha} \partial t^{m\alpha}} \right)_{s=s_0,t=t_0} \\ &= a\sum_{i=0}^{k} \sum_{j=0}^{m} U(k,m-j)V(k-i,m). \end{aligned}$$

Therefore, we get

$$F(k,m) = a\sum_{i=0}^{k} \sum_{j=0}^{m} U(k,m-j)V(k-i,m).$$

Theorem 3.3. *If*

$$f(s,t) = \frac{\partial^\alpha (au(s,t))}{\partial s^\alpha},$$

then

$$F(k,m) = \frac{a\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} U(k+1,m) \tag{13}$$

Proof. From Eq. (10), we get

$$\begin{aligned} F(k,m) &= \frac{1}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)} \left(\frac{\partial^{(k+m)\alpha} f(s,t)}{\partial s^{k\alpha} \partial t^{m\alpha}} \right)_{(s_0,t_0)} \\ &= \frac{a}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)} \left(\frac{\partial^{(k+m)\alpha} \left(\frac{\partial^\alpha u(s,t)}{\partial s^\alpha} \right)}{\partial s^{k\alpha} \partial t^{m\alpha}} \right)_{(s_0,t_0)} \end{aligned}$$

Table 1
Fractional PDE transform.

(k,m)	0	1	2	3	4
0	A	$\frac{A}{r\Gamma(\beta+1)}(c-\eta)$	$\frac{A}{r^2\Gamma(2\beta+1)}(c^2-2c\eta+\eta^2)$	$\frac{A}{r^3\Gamma(3\beta+1)}(c^3-3c^2\eta+3c\eta^2-\eta^3)$	$\frac{A}{r^4\Gamma(4\beta+1)}(c^4-4c^3\eta+6c^2\eta^2-4c\eta^3+\eta^4)$
1	$-cA$	$-\frac{A}{r\Gamma(\beta+1)}(c^2-c\eta)$	$-\frac{A}{r^2\Gamma(2\beta+1)}(c^3-2c^2\eta+c\eta^2)$	$-\frac{A}{r^3\Gamma(3\beta+1)}(c^4-3c^3\eta+3c^2\eta^2-c\eta^3)$	$-\frac{A}{r^4\Gamma(4\beta+1)}(c^5-4c^4\eta+6c^3\eta^2-4c^2\eta^3+c^2\eta^4)$
2	$\frac{c^2A}{2!}$	$\frac{A}{2!r\Gamma(\beta+1)}(c^3-c^2\eta)$	$\frac{A}{2!r^2\Gamma(2\beta+1)}(c^4-2c^3\eta+c^2\eta^2)$	$\frac{A}{2!r^3\Gamma(3\beta+1)}(c^5-3c^4\eta+3c^3\eta^2-c^2\eta^3)$	$\frac{A}{2!r^4\Gamma(4\beta+1)}(c^6-4c^5\eta+6c^4\eta^2-4c^3\eta^3+c^2\eta^4)$
3	$-\frac{c^3A}{3!}$	$-\frac{A}{3!r\Gamma(\beta+1)}(c^4-c^3\eta)$	$-\frac{A}{3!r^2\Gamma(2\beta+1)}(c^5-2c^4\eta+c^3\eta^2)$	$-\frac{A}{3!r^3\Gamma(3\beta+1)}(c^6-3c^5\eta+3c^4\eta^2-c^3\eta^3)$	$-\frac{A}{3!r^4\Gamma(4\beta+1)}(c^7-4c^6\eta+6c^5\eta^2-4c^4\eta^3+c^3\eta^4)$
4	$\frac{c^4A}{4!}$	$\frac{A}{4!r\Gamma(\beta+1)}(c^5-c^4\eta)$	$\frac{A}{4!r^2\Gamma(2\beta+1)}(c^6-2c^5\eta+c^4\eta^2)$	$\frac{A}{4!r^3\Gamma(3\beta+1)}(c^7-3c^6\eta+3c^5\eta^2-c^4\eta^3)$	$\frac{A}{4!r^4\Gamma(4\beta+1)}(c^8-4c^7\eta+6c^6\eta^2-4c^5\eta^3+c^4\eta^4)$

$$= \frac{a}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)}\left(\frac{\partial^{(k+m+1)\alpha}u(s,t)}{\partial s^{(k+1)\alpha}\partial t^{m\alpha}}\right)_{(s_0,t_0)}$$
$$= \frac{a\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)}U(k+1,m)$$

Thus,

$$F(k,m) = \frac{a\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)}U(k+1,m) \tag{14}$$

Theorem 3.4. *If*

$$f(s,t) = \frac{\partial^{\beta}(bu(s,t))}{\partial t^{\beta}},$$

then

$$F(k,m) = \frac{b\Gamma((k+1)\beta+1)}{\Gamma(1+k\beta)}U(k,m+1) \tag{15}$$

Proof. Similarly, by Eq. (10), we find that

$$F(k,m) = \frac{1}{\Gamma(k\beta+1)\Gamma(m\beta+1)}\left(\frac{\partial^{(k+m)\beta}f(s,t)}{\partial s^{k\beta}\partial t^{m\beta}}\right)_{(s_0,t_0)}$$
$$= \frac{b}{\Gamma(k\beta+1)\Gamma(m\beta+1)}\left(\frac{\partial^{(k+m)\beta}\left(\frac{\partial^{\beta}u(s,t)}{\partial t^{\beta}}\right)}{\partial s^{k\beta}\partial t^{m\beta}}\right)_{(s_0,t_0)}$$
$$= \frac{b}{\Gamma(k\beta+1)\Gamma(m\beta+1)}\left(\frac{\partial^{(k+m+1)\beta}u(s,t)}{\partial s^{k\beta}\partial t^{(m+1)\beta}}\right)_{(s_0,t_0)}$$
$$= \frac{b\Gamma((m+1)\beta+1)}{\Gamma(m\beta+1)}U(k,m+1)$$

Therefore,

$$F(k,m) = \frac{b\Gamma((k+1)\beta+1)}{\Gamma(1+k\beta)}U(k,m+1) \tag{16}$$

Theorem 3.5. . *If*

$$f(s,t) = \frac{\partial^{(l+r)\alpha}(cu(s,t))}{\partial s^{l\alpha}\partial t^{r\alpha}}$$

then

$$F(k,m) = \frac{c\Gamma((k+l)\alpha+1)\Gamma((m+r)\alpha+1)}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)}U(k+l,m+r). \tag{17}$$

where $l,r = 0,1,\ldots,n$ *and* $\alpha \in (0,1]$.

Proof. By Eq. (10), we have

$$F(k,m) = \frac{1}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)}\left(\frac{\partial^{(k+m)\alpha}f(s,t)}{\partial s^{k\alpha}\partial t^{m\alpha}}\right)_{(s_0,t_0)}$$
$$= \frac{c}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)}\left(\frac{\partial^{(k+m)\alpha}\left(\frac{\partial^{(l+r)\alpha}u(s,t)}{\partial s^{l\alpha}\partial t^{r\alpha}}\right)}{\partial s^{k\alpha}\partial t^{m\alpha}}\right)_{(s_0,t_0)}$$
$$= \frac{c}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)}\left(\frac{\partial^{(k+l+m+r)\alpha}u(x,y)}{\partial s^{(k+l)\alpha}\partial t^{(m+r)\alpha}}\right)_{(s_0,t_0)}$$
$$= \frac{c\Gamma((k+l)\alpha+1)\Gamma((m+r)\alpha+1)}{\Gamma(k\alpha+1)\Gamma(m\alpha+1)}U(k+l,m+r).$$

4. Fractional partial differential equation of growth model

In this section we provide the fractional growth model obtained from a modification of the McKendrick equation and pure growth model. Let $w(s,t)$ represents the density of population, μ denotes mortality and β is fractional order of model as well $g(s)$ denotes velocity. Then the fractional model is written as below.

$$\frac{\partial w(s,t)}{\partial t} + g(s)\frac{\partial^{\beta}w(s,t)}{\partial s^{\beta}} = -\mu w(s,t), \tag{18}$$

with $g(s) = \frac{1}{s}$, $s \in (0,1]$ and initial condition

$$w(0,t) = A(1-e^{-c(t-t_0)}).$$

Here A denotes the initial number of population and c is a growth rate. Furthermore, Eq. (18) is analyzed by generalized differential transform method to obtain the growth model. Therefore, by Definition 3.1, the transformation of the initial condition as follows.

$$w(0,m) = \frac{1}{m!}\left[\frac{d^m w(0,t)}{dt^m}\right]_{t=0} = \frac{(-c)^m}{m!}A.$$

Similarly, the transformation of fractional PDE is given as

$$W(k+1,m) = -\frac{(m+1)(\Gamma(k\beta+1))}{r\Gamma((k+1)\beta+1)}\left(W(k,m+1)+\mu W(k,m)\right).$$

with assuming $g(s) = \frac{1}{s} = r$. By running iteration we have, see Table 1.

If the iteration is proceeded further into m times, we obtain

$$w(s,t) = \left(1+\frac{s^{\beta}}{r\Gamma(\beta+1)}(c-\eta)\right)Ae^{-ct} + \left(\frac{s^{2\beta}}{r^2\Gamma(2\beta+1)}\left(c^2-2\eta c+\eta^2\right)\right)Ae^{-ct} + \left(\frac{s^{3\beta}}{r^3\Gamma(3\beta+1)}\left(c^3-3\eta c^2+3\eta^2 c+\eta^3\right)\right)Ae^{-ct}+\cdots$$

Therefore, the growth model for single population is given as

$$w(x,t) = Ae^{-ct}\sum_{m=0}^{n}\frac{(c-\eta)^m}{r^m\Gamma(m\beta+1)}s^{m\beta}. \tag{19}$$

Clearly, Eq. (19) is convergent for $s \in (0,1)$ and $c \geq 0$. For more detail the model will be examined on the abalone length growth in the next section.

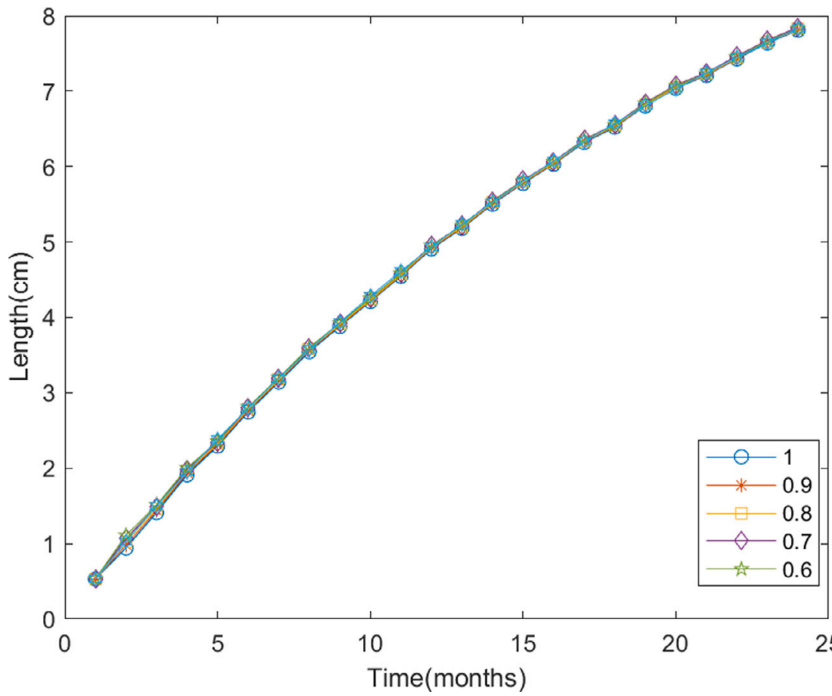

Fig. 1. Abalone length.

Table 2
Abalone length.

Month	l(cm)	Month	l(cm)	Month	l(cm)	Month	l(cm)
1	0.5322	7	3.1930	13	5.2365	19	6.8058
2	1.0258	8	3.5721	14	5.5276	20	7.0294
3	1.4982	9	3.9349	15	5.8062	21	7.2434
4	1.9503	10	4.2821	16	6.0728	22	7.4481
5	2.3829	11	4.6143	17	6.3279	23	7.6441
6	2.7968	12	4.9322	18	6.5722	24	7.8316

5. Applications

This part we provide the fractional growth model for option abalone length growth. Let consider the abalone length data in Lombok Marine Aquaculture Center Indonesia 2015[1] as in Table 2.

By processing the data with linear regression and Von Bertalanffy model, the growth of abalone length was presented in Eq. (1). Furthermore, According to the data and Eq. (18) we propose the following fractional equation. Let $l(s,t)$ represent the body length of abalone species at the s-age and time t. If $\mu(s,t) = \eta$ is an inhibition rate of abalone length growth and coefficient r is constant, the fractional equation is given as follows.

$$\frac{\partial l(s,t)}{\partial t} + r\frac{\partial^{\beta} l(s,t)}{\partial s^{\beta}} = -\eta l(s,t), \tag{20}$$

with initial condition

$$l(0,t) = l_i\left(1 - e^{-0.04305(t+0.0537509)}\right)$$

where $i = 0, 1, 2, 3, \ldots, 23$.

Using the GDTM concepts we may determine the transformation of the initial condition as follows.

$$L(0,m) = \frac{1}{m!}\left(\frac{d^m l(0,t)}{dt^m}\right)_{t=0} = \frac{(-0.04305)^m A}{m!}$$

where $A = l_i e^{-0.0023}$.

By applying the growth model (19), we find that the growth model for abalone length as follows.

$$l(s,t) = Ae^{-(0.04305)t}\sum_{m=0}^{23}\frac{(0.04305-\eta)^m}{r^m\Gamma(m\beta+1)}s^{m\beta}. \tag{21}$$

Table 3
Approximation of abalone length.

Month	r	η	l_1	$l_{0.9}$	$l_{0.8}$	$l_{0.7}$	$l_{0.6}$
1	–	–	0.5322	0.5322	0.5322	0.5322	0.5322
2	0.0490	0.0150	0.9409	0.9789	1.0198	1.0655	1.1076
3	0.0470	0.0280	1.4097	1.4357	1.4614	1.4897	1.5092
4	0.0450	0.0320	1.9108	1.9359	1.9601	1.9854	2.0024
5	0.0430	0.0360	2.2925	2.3109	2.3280	2.3476	2.3597
6	0.0420	0.0370	2.7458	2.7647	2.7821	2.8007	2.8101
7	0.0400	0.0383	3.1423	3.1596	3.1754	3.1964	3.2000
8	0.0380	0.0390	3.5440	3.5613	3.5769	3.5955	3.6010
9	0.0360	0.0400	3.8790	3.8937	3.9068	3.9252	3.9264
10	0.0350	0.0406	4.2105	4.2234	4.2349	4.2524	4.2517
11	0.0330	0.0410	4.5461	4.5583	4.5691	4.5886	4.5849
12	0.0320	0.0410	4.9083	4.9219	4.9340	4.9555	4.9517
13	0.0300	0.0415	5.1818	5.1933	5.2033	5.2236	5.2178
14	0.0280	0.0416	5.5021	5.5143	5.5250	5.5442	5.5404
15	0.0270	0.0418	5.7763	5.7876	5.7975	5.8191	5.8117
16	0.0260	0.0420	6.0316	6.0419	6.0508	6.0702	6.0635
17	0.0250	0.0420	6.3188	6.3300	6.3398	6.3597	6.3537
18	0.0230	0.0423	6.5227	6.5316	6.5393	6.5606	6.5501
19	0.0230	0.0422	6.8039	6.8144	6.8236	6.8449	6.8366
20	0.0210	0.0423	7.0371	7.0476	7.0567	7.0804	7.0696
21	0.0200	0.0425	7.2088	7.2170	7.2241	7.2449	7.2340
22	0.0200	0.0425	7.4282	7.4366	7.4440	7.4671	7.4542
23	0.0200	0.0425	7.6382	7.6469	7.6544	7.6781	7.6649
24	0.0190	0.0426	7.8093	7.8169	7.8235	7.8468	7.8326

Where l_{β} represents the abalone length with fractional order β.

Applying the model (21), we need to determine the parameter values of the model. Here r is a velocity, with $r = \triangle l : 10 \triangle t$. Meanwhile, the values of η are simulated refer to the data. The results of the model are indicated on the Table 3 as below.

From the Table 3 show that as velocity of the abalone length decreases the smaller the fractional order of the model, indicating the bigger prediction value. In addition, we find that the error of l_1, $l_{0.9}$, $l_{0.8}$, $l_{0.7}$, and $l_{0.6}$ are 0.041534; 0.029108; 0.020225; 0.021118; and 0.022219 respectively. Therefore, the best result of Eq. (21) is indicated by fractional order for $\beta = 0.8$. It can be also presented in Fig. 1 as below

On the other hand, Eq. (20) can be also expressed as the following system equations.

$$\frac{\partial l(s,t)}{\partial t} + r_1 \frac{\partial^\beta l(s,t)}{\partial s^\beta} = -\eta_1 l(s,t),$$
$$\frac{\partial l(s,t)}{\partial t} + r_2 \frac{\partial^\beta l(s,t)}{\partial s^\beta} = -\eta_2 l(s,t)$$
$$\vdots$$
$$\frac{\partial l(s,t)}{\partial t} + r_{23} \frac{\partial^\beta l(s,t)}{\partial s^\beta} = -\eta_{23} l(s,t)$$

or it can be written as

$$\begin{bmatrix} 1 & r_1 \\ 1 & r_2 \\ \vdots & \vdots \\ 1 & r_{23} \end{bmatrix} \begin{bmatrix} \frac{\partial l(s,t)}{\partial t} \\ \frac{\partial^\beta l(s,t)}{\partial s^\beta} \end{bmatrix} = \begin{bmatrix} -\eta_1 l(s,t) \\ -\eta_2 l(s,t) \\ \vdots \\ -\eta_{23} l(s,t) \end{bmatrix}$$

and

$$f(r_1, \eta_1, \beta) = l_1, f(r_2, \eta_2, \beta) = l_2, \ldots, f(r_{23}, \eta_{23}, \beta) = l_{23}.$$

6. Conclusion

This study shows that the modification of McKendrick equation into fractional partial differential equation with exponential function as the initial condition is capable of predicting the growth model specially in ecology. In this study, the fractional equation is analyzed by generalized differential transform method which produces the Taylor's series. By further substituting the parameter values of the series with the abalone length growth data, we obtained the approximations of abalone length growth. Furthermore, the growth model with fractional order $\beta = 0.8$ shows that the result is the most approaching to the real data compare to the other orders including integer order.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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