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Seidel Laplacian and Seidel Signless Laplacian Energies of Commuting Graph for Dihedral Groups

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Abstract In this paper, we discuss the energy of the commuting graph. The vertex set of the graph is dihedral groups and the edges between two distinct vertices represent the commutativity of the group elements. The spectrum of the graph is associated with the Seidel Laplacian and Seidel signless Laplacian matrices. The results are similar to the well-known fact that the energies are not odd integers. We also highlight the relation that the Seidel signless Laplacian energy is never less than Seidel Laplacian energy. Ultimately, we classify the graphs according to the energy value as the hyperenergetic.

Keywords: Seidel Laplacian matrix, Seidel signless Laplacian matrix, energy of a graph, commuting graph, dihedral groups.

Introduction

The commuting graph, represented by Γ_G and defined on the finite group G, has $G\backslash Z(G)$ as its vertex set. This graph requires that $v_p \neq v_q \in G\backslash Z(G)$ must be connected by an edge whenever $v_p v_q = v_q v_p$ [3]. An edge exists between v_p and v_q in Γ_G; such conditions are referred to as adjacent. The adjacency matrix of Γ_G is denoted as $A(\Gamma_G) = [a_{pq}]$, with a dimension of $n \times n$. If v_p and v_q are adjacent, a_{pq} equals 1, and if not, it equals 0. Furthermore, $P_{A(\Gamma_G)}(\lambda) = |\lambda I_n - A(\Gamma_G)|$ is the characteristic polynomial formula of $A(\Gamma_G)$, where I_n denotes the identity matrix with a dimension of $n \times n$ [4]. The eigenvalues of Γ_G, denoted as $\lambda_1, \lambda_2, \dots, \lambda_n$ are the roots of $P_{A(\Gamma_G)}(\lambda) = 0$. The collection of all $\lambda_1, \lambda_2, \dots, \lambda_n$ represented by $Spec(\Gamma_G) = \{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$ is referred to as the spectrum of Γ_G, with $k_1, k_2, \dots, k_n$ are the respective multiplicities of these values. The spectral radius of Γ_G is denoted by the formula $\rho(\Gamma_G) = max\{|\lambda|: \lambda \in Spec(\Gamma_G)\}$ [7]. Several scholarly articles examine the spectral radius and spectrum of alternative graph types, including the non-commuting graph [17] in relation to the Sombor matrix, the coprime graph [18], and the cubic power graph [12].

Gutman initially identified the adjacency energy of a finite graph in 1978 [6]. It is represented by $E_A(\Gamma_G)$ and defined as $E_A(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|$ for Γ_G with n vertices. Graphs consisting of n vertices and possessing an energy greater than $E_A(K_n)$ are deemed hyperenergetic, or equivalently, when $E_A(\Gamma_G)$ exceeds $2(n-1)$ [8]. Furthermore, energy values are never odd integers ([2],[9]).

A new graph matrix definition was put forward by Van Lint & Seidel (1966) [19], named the Seidel matrix of Γ_G, denoted by $S(\Gamma_G) = [s_{pq}]$ whose $(p,q)-$th entry is

$$s_{pq} = \begin{cases} -1, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 1, & \text{if } v_p \neq v_q \text{ and they are not adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The diagonal degree matrix of order $n \times n$ associated with Γ_G is given by $D(\Gamma_G) = diag\,[n-1-2d_{11}, n-1-2\,d_{22}, \cdots, n-1-d_{nn}\,]$, where d_{ii} is the degree of vertex v_i for $i = 1,2,\dots,n$. The Seidel

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Laplacian matrix [10] of order $n \times n$ associated with Γ_G is

$$SL(\Gamma_G) = D(\Gamma_G) - S(\Gamma_G).$$

The Seidel signless Laplacian matrix [11] of order $n \times n$ associated with Γ_G is

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G).$$

Our study centers on the Seidel Laplacian (SL) and Seidel signless Laplacian (SSL) matrices of Γ_G pertaining to the non-abelian dihedral groups of order $2n$, $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$, where n is more than or equal to three. Furthermore, its elements can be represented as a^i and $a^i b$ [1]. For odd n, the center of D_{2n}, denoted by $Z(D_{2n})$, is equivalent to the set $\{e\}$; for even n, it is equal to $\{e, a^{\frac{n}{2}}\}$. The centralizer of a^i in D_{2n} is denoted by $C_{D_{2n}}(a^i) = \{ a^j : 1 \leq j \leq n \}$, and for $a^i b$, if n is odd, it is $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, and if n is even, it is $C_{D_{2n}}(a^i b) = \{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}$.

Some researchers have published recent findings regarding the energy of Γ_G for D_{2n}, where $n \geq 3$. Degree exponent sum [13], maximum and minimum degree [14], degree subtraction [15], and neighbor degree sum [16] matrices were among the graph matrices they performed. As an extension of those investigations, the spectral radius and energy of Γ_G for D_{2n} corresponding with Seidel Laplacian and Seidel signless Laplacian matrices are discussed in this paper. The methodology involves the following steps: generate the Seidel Laplacian and Seidel signless Laplacian matrices of Γ_G, determine its eigenvalues and spectrum, examine $\rho(\Gamma_G)$, calculate the Seidel Laplacian and Seidel signless Laplacian energies, and subsequently observe the correlation between $\rho(\Gamma_G)$ and the obtained energies. Additionally, we examine the hyperenergetic characteristic of Γ_G.

Preliminaries

We investigate the commuting graph for the subset of dihedral groups of order $2n$, D_{2n} denoted by Γ_G, where G is one of the following values: G_1, G_2 or $G_1 \cup G_2$. The set G_1 is defined as $\{a^i : 1 \leq i \leq n\} \setminus Z(D_{2n})$, and G_2 is defined as $\{a^i b : 1 \leq i \leq n\}$. We define the Seidel Laplacian energy of Γ_G as

$$E_{SL}(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of $SL(\Gamma_G)$, which need not be distinct from one another. The SL-spectral radius of Γ_G can be calculated as

$$\rho_{SL}(\Gamma_G) = max\{|\lambda| : \lambda \in Spec(\Gamma_G)\}.$$

Furthermore, in the case where the SL-energy fulfills the subsequent criteria, Γ_G associated with the SL-matrix can be categorized as a hyperenergetic graph, given that it consists of $2n-1$ (odd n) and $2n-2$ (even n) vertices,

$$E_{SL}(\Gamma_G) > \begin{cases} 4(n-1), & \text{for odd } n \\ 4(n-1)-2, & \text{for even } n. \end{cases}$$

In order to determine the roots of $P_{SL(\Gamma_G)}(\lambda) = 0$, elementary row and column operations must be performed on $P_{SL(\Gamma_G)}(\lambda)$. Denote R_i and R_i' as the i-th and new i-th rows, respectively, that result from the row operation of $P_{SL(\Gamma_G)}(\lambda)$. Furthermore, designate the i-th column as C_i, and denote the new i-th column obtained from a column operation of $P_{SL(\Gamma_G)}(\lambda)$ as C_i'. The above notations also can be applied to the Seidel signless Laplacian matrix.

Now we are moving to the properties for constructing the SL and SSL-matrices. Some previous results of Γ_G are given below:

Theorem 2.1. [13] Let Γ_G be the commuting graph for D_{2n}, where $G = G_1 \cup G_2$. Then

1. The degree of a^i on Γ_G is $d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd} \\ n-3, & \text{if } n \text{ is even,} \end{cases}$
2. the degree of $a^i b$ on Γ_G is $d_{a^i b} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.2. [13] Let Γ_G be the commuting graph for D_{2n}.

1. If $G = G_1$, then $\Gamma_G \cong K_m$, where $m = |G_1|$.
2. If $G = G_2$, then $\Gamma_G \cong \begin{cases} \overline{K}_n, & \text{if } n \text{ is odd} \\ 1-\text{regular graph}, & \text{if } n \text{ is even.} \end{cases}$

By applying these two theorems, the characteristic polynomial of Γ_G can be ascertained through the construction of its SL and SSL-matrices. The subsequent finding offers assistance in streamlining the procedure for deriving the characteristic polynomial of Γ_G for D_{2n}.

Theorem 2.3. [5] If a square matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ consists of four blocks, where $|A| \neq 0$, then

$$|M| = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.$$

Main Results

The following theorem gives the characteristic polynomial of some matrices.

Theorem 3.1. If s, t are real numbers, then the characteristic polynomial of an $m \times m$ matrix

$$M = \begin{bmatrix} s & t & \cdots & t \\ t & s & \cdots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & s \end{bmatrix}$$

can be simplified in an expression as

$$P_M(\lambda) = (\lambda - s - (m-1)t)(\lambda - s + t)^{m-1}.$$

Proof.
The characteristic polynomial of M is $P_M(\lambda) = |(\lambda - s + t)I_m + tJ_m|$. We replace R_i by $R_i' = R_i - R_1$, for every $2 \leq i \leq m$. Then we see

$$P_M(\lambda) = \begin{vmatrix} \lambda - s & -tJ_{1\times(m-1)} \\ -(\lambda - s + t)J_{(m-1)\times 1} & (\lambda - s + t)I_{m-1} \end{vmatrix}.$$

We replace C_1 by $C_1' = C_1 + C_2 + \cdots + C_m$, then

$$P_M(\lambda) = \begin{vmatrix} \lambda - s - (m-1)t & -tJ_{1\times(m-1)} \\ 0_{(m-1)\times 1} & (\lambda - s + t)I_{m-1} \end{vmatrix}.$$

It is obvious that $P_M(\lambda)$ is an upper triangular matrix. Thus, it can be simplified as the product of the main diagonal entries as given below:

$$P_{M(\Gamma_G)}(\lambda) = (\lambda - s - (m-1)t)(\lambda - s + t)^{m-1}.$$

□

Theorem 3.2. If s, t are real numbers, and even number n, then the characteristic polynomial of an $n \times n$ matrix

$$M = \begin{bmatrix} s & t & \cdots & t & -t & t & \cdots & t \\ t & s & \cdots & t & t & -t & \cdots & t \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & s & t & t & \cdots & -t \\ -t & t & \cdots & t & s & t & \cdots & t \\ t & -t & \cdots & t & t & s & \cdots & t \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & -t & t & t & \cdots & s \end{bmatrix}$$

can be simplified in an expression as

$$P_M(\lambda) = (\lambda - s + (3-n)t)(\lambda - s + 3t)^{\frac{n}{2}-1}(\lambda - s - t)^{\frac{n}{2}}.$$

Proof.
Let M be a square matrix of size $n \times n$ as follows:

$$M = \begin{bmatrix} (s-t)I_{\frac{n}{2}} + tJ_{\frac{n}{2}} & -2tI_{\frac{n}{2}} + tJ_{\frac{n}{2}} \\ -2tI_{\frac{n}{2}} + tJ_{\frac{n}{2}} & (s-t)I_{\frac{n}{2}} + tJ_{\frac{n}{2}} \end{bmatrix}.$$

We derive the following determinant

$$P_{M(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda - s + t)I_{\frac{n}{2}} - tJ_{\frac{n}{2}} & 2tI_{\frac{n}{2}} - tJ_{\frac{n}{2}} \\ 2tI_{\frac{n}{2}} - tJ_{\frac{n}{2}} & (\lambda - s + t)I_{\frac{n}{2}} - tJ_{\frac{n}{2}} \end{vmatrix}.$$

To begin, we replace $R_{\frac{n}{2}+i}$ by $R_{\frac{n}{2}+i}' = R_{\frac{n}{2}+i} - R_i$, for $1 \leq i \leq \frac{n}{2}$. Then, $P_{M(\Gamma_G)}(\lambda)$ can be expressed as

$$P_{M(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda - s + t)I_{\frac{n}{2}} - tJ_{\frac{n}{2}} & 2tI_{\frac{n}{2}} - tJ_{\frac{n}{2}} \\ -(\lambda - s - t)I_{\frac{n}{2}} & (\lambda - s - t)I_{\frac{n}{2}} \end{vmatrix}.$$

Consequently, we replace C_i by $C_i' = C_i + C_{\frac{n}{2}+i}$, for every $1 \le i \le \frac{n}{2}$. Then $P_{M(\Gamma_G)}(\lambda)$ can be written as

$$P_{M(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda - s + 3t)I_{\frac{n}{2}} - 2tJ_{\frac{n}{2}} & 2tI_{\frac{n}{2}} - tJ_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda - s - t)I_{\frac{n}{2}} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

By using Theorem 2.3, it is the form of

$$P_{M(\Gamma_G)}(\lambda) = |A||D - CA^{-1}B| = |A||D|, \tag{1}$$

since $C = 0$. The next step for $|A|$, following Theorem 3.1 with $s = s - t$, $t = 2t$, and $m = \frac{n}{2}$, consequently

$$|A| = (\lambda - s + (3 - n)t)(\lambda - s + 3t)^{\frac{n}{2}-1}. \tag{2}$$

Meanwhile $D = (\lambda - s - t)I_{\frac{n}{2}}$ and this is a diagonal matrix. Then

$$|D| = (\lambda - s - t)^{\frac{n}{2}}. \tag{3}$$

Now we substitute Equations (2) and (3) to Equation (1), therefore,

$$P_{M(\Gamma_G)}(\lambda) = (\lambda - s + (3 - n)t)(\lambda - s + 3t)^{\frac{n}{2}-1}(\lambda - s - t)^{\frac{n}{2}}.$$

□

Theorem 3.3. If s, t are real numbers, and odd number n then the characteristic polynomial of an $(2n - 1) \times (2n - 1)$ matrix

$$M = \begin{bmatrix} s & t & \cdots & t & -t & -t & \cdots & -t \\ t & s & \cdots & t & -t & -t & \cdots & -t \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & s & -t & -t & \cdots & -t \\ -t & -t & \cdots & -t & u & -t & \cdots & -t \\ -t & -t & \cdots & -t & -t & u & \cdots & -t \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -t & -t & \cdots & -t & -t & -t & \cdots & u \end{bmatrix} = \begin{bmatrix} (s - t)I_{n-1} + tJ_{n-1} & -tJ_{(n-1)\times n} \\ -tJ_{n\times(n-1)} & (u + t)I_n - tJ_n \end{bmatrix}.$$

can be simplified in an expression as

$$P_M(\lambda) = (\lambda - s + t)^{n-2}\big((\lambda - u + (n - 1)t)(\lambda - s - (n - 2)t) - n(n - 1)t^2\big)(\lambda - t - u)^{n-1}.$$

Proof.
The determinant below is the characteristic polynomial for M,

$$P_M(\lambda) = \begin{vmatrix} (\lambda - s + t)I_{n-1} - tJ_{n-1} & tJ_{(n-1)\times n} \\ t\,J_{n\times(n-1)} & (\lambda - u + t)I_n + tJ_n \end{vmatrix}.$$

To begin, we replace R_{1+i} by $R_{1+i}' = R_{i+1} - R_1$ for $1 \le i \le n - 2$ and replace R_{n+i} by $R_{n+i}' = R_{n+i} - R_n$, for $1 \le i \le n - 1$. Then, $P_M(\lambda)$ can be expressed as

$$P_M(\lambda) = \begin{vmatrix} \lambda - s & -tJ_{1\times(n-2)} & t & tJ_{1\times(n-1)} \\ -(\lambda - t - s)J_{(n-2)\times 1} & (\lambda - s - t)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ t & tJ_{1\times(n-2)} & \lambda - u & tJ_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{n-1} & -(\lambda - t - u)J_{(n-1)\times 1} & (\lambda - t - u)I_{n-1} \end{vmatrix}.$$

Consequently, we replace C_1 by $C_1' = C_1 + C_2 + \cdots + C_{n-1}$ and replace C_n by $C_n' = C_{n+1} + C_{n+2} + \cdots + C_{2n-1}$. Then $P_M(\lambda)$ can be written as

$$P_M(\lambda) = \begin{vmatrix} \lambda - s - (n - 2)t & -tJ_{1\times(n-2)} & nt & tJ_{1\times(n-1)} \\ 0_{(n-2)\times 1} & (\lambda - s + t)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ (n - 1)t & tJ_{1\times(n-2)} & \lambda - u + (n - 1)t & tJ_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda - t - u)I_{n-1} \end{vmatrix}.$$

We replace R_n by $R_n' = R_n + \left(\frac{(1-n)t}{\lambda - s - (n-2)t}\right)R_1$ and following by $R_n' = R_n + \left(\frac{(1-n)t}{(\lambda - s + t)(\lambda - s - (n-2)t)}\right)R_2 +$

$\left(\frac{(1-n)t}{(\lambda-s+t)(\lambda-s-(n-2)t)}\right)R_3+\cdots+\left(\frac{(1-n)t}{(\lambda-s+t)(\lambda-s-(n-2)t)}\right)R_{n-1}$, consequently, $P_M(\lambda)$ can be expressed as

$$P_M(\lambda)=\begin{vmatrix}\lambda-s-(n-2)t & -tJ_{1\times(n-2)} & nt & tJ_{1\times(n-1)}\\ 0_{(n-2)\times 1} & (\lambda-s+t)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)}\\ 0 & 0_{1\times(n-2)} & \frac{(\lambda-u+(n-1)t)(\lambda-s-(n-2)t)+n(1-n)t^2}{\lambda-s-(n-2)t} & \frac{(1-n)t}{(\lambda-s+t)(\lambda-s-(n-2)t)}J_{1\times(n-1)}\\ 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-t-u)I_{n-1}\end{vmatrix}.$$

It is obvious that $P_M(\lambda)$ is a diagonal matrix as follows:

$$P_M(\lambda)=(\lambda-s+t)^{n-2}\big((\lambda-u+(n-1)t)(\lambda-s-(n-2)t)-n(n-1)t^2\big)(\lambda-t-u)^{n-1}.$$

□

Theorem 3.4. If r,s,t are real numbers, and even number n, then the characteristic polynomial of an $(2n-2)\times(2n-2)$ matrix

$$M=\begin{bmatrix} r & \cdots & t & -t & \cdots & -t & -t & \cdots & -t\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ t & \cdots & r & -t & \cdots & -t & -t & \cdots & -t\\ -t & \cdots & -t & s & \cdots & -t & t & \cdots & -t\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ -t & \cdots & -t & -t & \cdots & s & -t & \cdots & t\\ -t & \cdots & -t & t & \cdots & -t & s & \cdots & -t\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ -t & \cdots & -t & -t & \cdots & t & -t & \cdots & s\end{bmatrix}=\begin{bmatrix}(r-t)I_{n-2}+tJ_{n-2} & -tJ_{(n-2)\times\frac{n}{2}} & -tJ_{(n-2)\times\frac{n}{2}}\\ -tJ_{\frac{n}{2}\times(n-2)} & (s+t)I_{\frac{n}{2}}-tJ_{\frac{n}{2}} & 2tI_{\frac{n}{2}}-tJ_{\frac{n}{2}}\\ -tJ_{\frac{n}{2}\times(n-2)} & 2tI_{\frac{n}{2}}-tJ_{\frac{n}{2}} & (s+t)I_{\frac{n}{2}}-tJ_{\frac{n}{2}}\end{bmatrix}.$$

can be simplified in an expression as

$$P_M(\lambda)=(\lambda+t-r)^{n-3}(\lambda+t-s)^{\frac{n}{2}}(\lambda-3t-s)^{\frac{n}{2}-1}(\lambda^2-(s+r)\lambda+rs-(n-3)(r-s)t-(2n^2-8n+9)t^2).$$

Proof.

The determinant below is the characteristic polynomial for M,

$$P_M(\lambda)=\begin{vmatrix}(\lambda-r+t)I_{n-2}-tJ_{n-2} & tJ_{(n-2)\times\frac{n}{2}} & tJ_{(n-2)\times\frac{n}{2}}\\ tJ_{\frac{n}{2}\times(n-2)} & (\lambda-s-t)I_{\frac{n}{2}}+tJ_{\frac{n}{2}} & -2tI_{\frac{n}{2}}+tJ_{\frac{n}{2}}\\ tJ_{\frac{n}{2}\times(n-2)} & -2tI_{\frac{n}{2}}+tJ_{\frac{n}{2}} & (\lambda-s-t)I_{\frac{n}{2}}+tJ_{\frac{n}{2}}\end{vmatrix}.$$

To begin, we replace $R_{n-2+\frac{n}{2}+i}$ by $R'_{n-2+\frac{n}{2}+i}=R_{n-2+\frac{n}{2}+i}-R_{n-2+i}$, for every $1\le i\le\frac{n}{2}$, following by replacing C_{n-2+i} with $C'_{n-2+i}=C_{n-2+i}+C_{n-2+\frac{n}{2}+i}$, replacing R_{n-1+i} with $R'_{n-1+i}=R_{n-1+i}-R_{n-1}$, for every $1\le i\le\frac{n}{2}-1$, and replacing C_{n-1} with $C'_{n-1}=C_{n-1}+C_n+C_{n+1}+\cdots+C_{n-2+\frac{n}{2}}$. Then, $P_M(\lambda)$ can be expressed as

$$P_M(\lambda)=\begin{vmatrix}\lambda-r & -tJ_{1\times(n-3)} & nt & 2tJ_{1\times\left(\frac{n}{2}-1\right)} & t & tJ_{1\times\left(\frac{n}{2}-1\right)}\\ -tJ_{(n-3)\times 1} & (\lambda-r+t)I_{n-3}-tJ_{n-3} & ntJ_{(n-3)\times 1} & 2tJ_{(n-3)\times\left(\frac{n}{2}-1\right)} & tJ_{(n-3)\times 1} & tJ_{(n-3)\times\left(\frac{n}{2}-1\right)}\\ t & tJ_{1\times(n-3)} & \lambda-s+(n-3)t & 2tJ_{1\times\left(\frac{n}{2}-1\right)} & -t & tJ_{1\times\left(\frac{n}{2}-1\right)}\\ 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\frac{n}{2}-1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & (\lambda-3t-s)I_{\frac{n}{2}-1} & 2tJ_{\left(\frac{n}{2}-1\right)\times 1} & -2tI_{\frac{n}{2}-1}\\ 0 & 0_{1\times(n-3)} & 0 & 0_{1\times\left(\frac{n}{2}-1\right)} & \lambda+t-s & 0_{1\times\left(\frac{n}{2}-1\right)}\\ 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\frac{n}{2}-1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\frac{n}{2}-1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & (\lambda+t-s)I_{\frac{n}{2}-1}\end{vmatrix}.$$

Consequently, we replace C_i by $C'_i=C_i-C_{n-2}$ for every $1\le i\le n-3$, and replace R_{n-2} by $R'_{n-2}=R_{n-2}+R_1+R_2+\cdots+R_{n-3}$. Then $P_M(\lambda)$ can be written as

$$P_M(\lambda) = \begin{vmatrix} (\lambda - r + t)I_{n-3} & -tJ_{(n-3)\times 1} & ntJ_{(n-3)\times 1} & 2tJ_{(n-3)\times(\frac{n}{2}-1)} & tJ_{(n-3)\times 1} & tJ_{(n-3)\times(\frac{n}{2}-1)} \\ 0_{1\times(n-3)} & \lambda - r - (n-3)t & n(n-2)t & 2(n-2)tJ_{1\times(\frac{n}{2}-1)} & (n-2)t & (n-2)tJ_{1\times(\frac{n}{2}-1)} \\ 0_{1\times(n-3)} & t & \lambda - s + (n-3)t & 2tJ_{1\times(\frac{n}{2}-1)} & -t & tJ_{1\times(\frac{n}{2}-1)} \\ 0_{(\frac{n}{2}-1)\times(n-3)} & 0_{(\frac{n}{2}-1)\times 1} & 0_{(\frac{n}{2}-1)\times 1} & (\lambda - 3t - s)I_{\frac{n}{2}-1} & 2tJ_{(\frac{n}{2}-1)\times 1} & -2tI_{\frac{n}{2}-1} \\ 0_{1\times(n-3)} & 0 & 0 & 0_{1\times(\frac{n}{2}-1)} & \lambda + t - s & 0_{1\times(\frac{n}{2}-1)} \\ 0_{(\frac{n}{2}-1)\times(n-3)} & 0_{(\frac{n}{2}-1)\times 1} & 0_{(\frac{n}{2}-1)\times 1} & 0_{\frac{n}{2}-1} & 0_{(\frac{n}{2}-1)\times 1} & (\lambda + t - s)I_{\frac{n}{2}-1} \end{vmatrix}.$$

Based on Theorem 2.3, it implies that $P_M(\lambda)$ can be expressed as

$$\begin{aligned} P_M(\lambda) &= (\lambda + t - r)^{n-3}(\lambda + t - s)^{\frac{n}{2}}(\lambda - 3t - s)^{\frac{n}{2}-1}\left((\lambda - r - (n-3)t)(\lambda - s + (n-3)t) - n(n-2)t^2\right) \\ &= (\lambda + t - r)^{n-3}(\lambda + t - s)^{\frac{n}{2}}(\lambda - 3t - s)^{\frac{n}{2}-1}(\lambda^2 - (s+r)\lambda + rs - (n-3)(r-s)t - (2n^2 - 8n + 9)t^2). \end{aligned}$$

□

The following theorem is the result of Seidel Laplacian energy of the commuting graph Γ_G, where $G = G_1$ or $G = G_2$.

Theorem 3.5. Let Γ_G be the commuting graph for D_{2n}, and $E_{SL}(\Gamma_G)$ be the Seidel Laplacian energy of Γ_G. If $G = G_1$ or $G = G_2$, then

1. If $G = G_1$, then $E_{SL}(\Gamma_G) = \begin{cases} (n-2)(n-1), & \text{if } n \text{ is odd} \\ (n-3)(n-2), & \text{if } n \text{ is even.} \end{cases}$
2. If $G = G_2$, then $E_{SL}(\Gamma_G) = \begin{cases} n(n-1), & \text{if } n \text{ is odd} \\ n(n-3), & \text{if } n \text{ is even.} \end{cases}$

Proof.
1. For $G = G_1$ and n is odd, from Theorem 2.2 (1), we know that $\Gamma_G \cong K_m$, where $m = |G_1| = n - 1$, and from Theorem 2.1 (1) clearly that every vertex of Γ_G has degree $n - 2$. Then $D(\Gamma_G)$ is an $(n-1) \times (n-1)$ diagonal matrix whose diagonal entries are $n - 1 - 1 - 2(n-2) = 2 - n$ as $diag(2-n, 2-n, \cdots, 2-n)$. The Seidel Laplacian matrix of Γ_G is

$$SL(\Gamma_G) = D(\Gamma_G) - S(\Gamma_G) = \begin{bmatrix} 2-n & 0 & \cdots & 0 \\ 0 & 2-n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2-n \end{bmatrix} - \begin{bmatrix} 0 & -1 & \cdots & -1 \\ -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 2-n & 1 & \cdots & 1 \\ 1 & 2-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2-n \end{bmatrix}.$$

Following Theorem 3.1 with $s = 2 - n$, $t = 1$, and $m = n - 1$, then the characteristic polynomial of $SL(\Gamma_G)$

$$P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda + n - 1)^{n-2}.$$

Hence, the roots of $P_{SL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = 0$ and $\lambda_2 = 1 - n$ with multiplicity $n - 2$. Consequently, the spectrum of Γ_G is

$$Spec(\Gamma_G) = \{(0)^1, (1-n)^{n-2}\}.$$

The Seidel Laplacian energy of Γ_G will be

$$E_{SL}(\Gamma_G) = (1)|0| + (n-2)|1-n| = (n-2)(n-1).$$

Same idea for $G = G_1$ and n is even, from Theorem 2.2 (1), we know that $\Gamma_G \cong K_m$, where $m = |G_1| = n - 2$, and from Theorem 2.1 (1) clearly that every vertex of Γ_G has degree $n - 3$. Then $D(\Gamma_G)$ is an $(n-2) \times (n-2)$ diagonal matrix whose diagonal entries are $n - 2 - 1 - 2(n-3) = 3 - n$ as as $diag(3-n, 3-n, \cdots, 3-n)$. The Seidel Laplacian matrix of Γ_G is

$$SL(\Gamma_G) = D(\Gamma_G) - S(\Gamma_G) = \begin{bmatrix} 3-n & 1 & \cdots & 1 \\ 1 & 3-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 3-n \end{bmatrix} = [(2-n)I_{n-2} + J_{n-2}].$$

Following Theorem 3.1 with $s = 3 - n$, $t = 1$, and $m = n - 2$, we get the characteristic polynomial of $SL(\Gamma_G)$

$$P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda + n - 2)^{n-3}.$$

The roots of $P_{SL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = 0$ and $\lambda_2 = 2 - n$ with multiplicity $n - 3$. Thus, the spectrum of Γ_G is

$$Spec(\Gamma_G) = \{(1)^1, (2-n)^{n-3}\}.$$

Therefore, the Seidel Laplacian energy of Γ_G will be

$$E_{SL}(\Gamma_G) = (1)|0| + (n-3)|2-n| = (n-3)(n-2).$$

2. **When n is odd**. Based on Theorem 2.2 (2), $\Gamma_G \cong \overline{K}_n$, for $G = G_2$ which clearly shows that all of the vertices have degree zero. Then $D(\Gamma_G)$ is an $n \times n$ diagonal matrix whose diagonal entries are $n-1-2(0) = n-1$ as as $diag(n-1, n-1, \cdots, n-1)$. Then an $n \times n$ Seidel Laplacian matrix of Γ_G is

$$SL(\Gamma_G) = D(\Gamma_G) - S(\Gamma_G) = \begin{bmatrix} n-1 & 0 & \cdots & 0 \\ 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

Following Theorem 3.1 with $s = n-1$, $t = -1$, and $m = n$, then the characteristic polynomial of $SL(\Gamma_G)$

$$P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda - n)^{n-1}.$$

Then the roots of $P_{SL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = 0$ and $\lambda_2 = n$ with multiplicity $n-1$. Thus, the spectrum of Γ_G is

$$Spec(\Gamma_G) = \{(n)^{n-1}, (0)^1\}.$$

The Seidel Laplacian energy of Γ_G is

$$E_{SL}(\Gamma_G) = (1)|0| + (n-1)|n| = n(n-1).$$

When n is even. According to theorem 2.2 (2), for $G = G_2, \Gamma_G$ is a regular graph with degree one, or in other words, the edges only connect between $a^i b$ and $a^{\frac{n}{2}+i} b$. Then $D(\Gamma_G)$ is an $n \times n$ diagonal matrix whose diagonal entries are $n-1-2(1) = n-3$ as $diag(n-3, n-3, \cdots, n-3)$. Then an $n \times n$ Seidel Laplacian matrix of Γ_G is

$$SL(\Gamma_G) = \left[\begin{array}{cccc|cccc} n-3 & -1 & \cdots & -1 & 1 & -1 & \cdots & -1 \\ -1 & n-3 & \cdots & -1 & -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-3 & -1 & -1 & \cdots & 1 \\ \hline 1 & -1 & \cdots & -1 & n-3 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & -1 & -1 & n-3 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 & -1 & -1 & \cdots & n-3 \end{array}\right] = \begin{bmatrix} (n-2)I_{\frac{n}{2}} - J_{\frac{n}{2}} & 2I_{\frac{n}{2}} - J_{\frac{n}{2}} \\ 2I_{\frac{n}{2}} - J_{\frac{n}{2}} & (n-2)I_{\frac{n}{2}} - J_{\frac{n}{2}} \end{bmatrix}.$$

Following Theorem 3.2 with $s = n-3$ and $t = -1$, therefore,

$$P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda - n)^{\frac{n}{2}-1}(\lambda + 4 - n)^{\frac{n}{2}}.$$

Then the roots of $P_{SL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = 0$, $\lambda_2 = n$ with multiplicity $\frac{n}{2} - 1$, and $\lambda_3 = n-4$ with multiplicity $\frac{n}{2}$. Thus, the spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{(n)^{\frac{n}{2}-1}, (n-4)^{\frac{n}{2}}, (0)^1\right\}.$$

and the Seidel Laplacian energy of Γ_G is

$$E_{SL}(\Gamma_G) = (1)|0| + \left(\frac{n}{2} - 1\right)|n| + \left(\frac{n}{2}\right)|n-4| = n(n-3).$$

□

We now formulate $P_{SL(\Gamma_G)}(\lambda)$ and calculate the Seidel Laplacian energy of Γ_G for $G = G_1 \cup G_2$. The following Theorem gives the spectrum, SL-spectral radius, and SL-energy of Γ_G for $G = G_1 \cup G_2$. Then, the relation between them is obtained at the end of this paper.

Theorem 3.6. Let Γ_G be the commuting graph for D_{2n}, where $G = G_1 \cup G_2 \subset D_{2n}$, then the SL-energy for Γ_G is

1. for odd n, $E_{SL}(\Gamma_G) = 2(n+1)(n-1)$,
2. for even n, $E_{SL}(\Gamma_G) = 2(n^2 - n - 3)$.

Proof.

1. For the case of odd n, we know that $Z(D_{2n}) = \{e\}$ which implies that there are $2n-1$ vertices for Γ_G. From Theorem 2.2, the degree of $a^i \in G$, $d_{a^i} = n-2$ and the degree of $a^i b \in G$, $d_{a^i b} = 0$, for all $1 \leq$

$i \leq n$. Then $D(\Gamma_G)$ is a $(2n-1) \times (2n-1)$ diagonal matrix whose diagonal entries are $2n-1-1-2(n-2) = 2$ for element a^i, and are $2n-1-1-2(0) = 2(n-1)$, for element $a^i b$ as

$$diag(2, 2, \cdots, 2, 2(n-1), 2(n-1), \cdots, 2(n-1)).$$

From the fact that the centralizer of a^i in D_{2n} is $\{e, a, a^2, \cdots, a^{n-1}\}$, then the vertex a^i, for $1 \leq i \leq n-1$, is adjacent to all vertices of G_1, however, it is not adjacent to all vertices of G_2. While the centralizer of $a^i b$ in D_{2n} is $\{e, a^i b\}$ implies that for $1 \leq i \leq n$, vertex $a^i b$ is not connected with all other elements of $G_1 \cup G_2$. Then a $(2n-1) \times (2n-1)$ Seidel Laplacian matrix of Γ_G is

$$SL(\Gamma_G) = D(\Gamma_G) - S(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccc|cccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{cccc|cccc} 2 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ 1 & 2 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 & -1 & -1 & \cdots & -1 \\ \hline -1 & -1 & \cdots & -1 & 2(n-1) & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & 2(n-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 2(n-1) \end{array}\right] \end{array}.$$

Then by using Theorem 3.3, with $s = 2$, $u = 2(n-1)$, $t = 1$, $n_1 = n-1$ and $n_2 = n$, we obtain

$$P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda-1)^{n-2}(\lambda-2n+1)^n.$$

This result is the four eigenvalues obtained from $P_{SL(\Gamma_G)}(\lambda)$. They are as single $\lambda_1 = 0$, $\lambda_2 = 1$ of multiplicity $n-2$ and $\lambda_3 = 2n-1$ of multiplicity n. Hence, the SL –spectrum for Γ_G is as follows

$$Spec(\Gamma_G) = \{(2n-1)^n, (1)^{n-2}, (0)^1\}.$$

Now for $i = 1,2,3$, the maximum of absolute eigenvalues $|\lambda_i|$ is the SL-spectral radius of Γ_G,

$$\rho_{SL}(\Gamma_G) = 2n-1.$$

By computing the eigenvalues from $Spec(\Gamma_G)$, then the SL-energy for Γ_G is

$$E_{SL}(\Gamma_G) = (n)|2n-1| + (n-2)|1| + (1)|0| = 2(n+1)(n-1).$$

2. Suppose now n is even. Since $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, Γ_G, where $G = G_1 \cup G_2$ has $2n-2$ vertices with $n-2$ vertices from a^i, for $1 \leq i < \frac{n}{2}$, $\frac{n}{2} < i < n$, and n vertices from $a^i b$, for $1 \leq i \leq n$. Using Theorem 2.2, we know that $d_{a^i} = n-3$ and $d_{a^i b} = 1$, for all $1 \leq i \leq n$, then $D(\Gamma_G)$ is a $(2n-2) \times (2n-2)$ diagonal matrix whose diagonal entries are $2n-2-1-2(n-3) = 3$ for element a^i, and are $2n-2-1-2(1) = 2n-5$, for element $a^i b$ as

$$diag[3, 3, \cdots, 3, 2n-5, 2n-5, \cdots, 2n-5].$$

Again, considering the centralizer of a^i in D_{2n}, then all the members of G_1 are only connected with the elements of G_1. Since the centralizer of $a^i b$ is $\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}$, then two vertices $a^i b$ and $a^{\frac{n}{2}+i} b$ are always connected in Γ_G, which implies a $(2n-2) \times (2n-2)$ Seidel Laplacian matrix of Γ_G is $SL(\Gamma_G) = D(\Gamma_G) - S(\Gamma_G)$ as follows:

$$SL(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^n \\ b \\ ab \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ a^{\frac{n}{2}+1}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccc|cccc|cccc} a & a^2 & \cdots & a^n & b & ab & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{cccc|cccc|cccc} 3 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ 1 & 3 & \cdots & 1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 3 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ \hline -1 & -1 & \cdots & -1 & 2n-5 & -1 & \cdots & -1 & 1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & 2n-5 & \cdots & -1 & -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 2n-5 & -1 & -1 & \cdots & 1 \\ \hline -1 & -1 & \cdots & -1 & 1 & -1 & \cdots & -1 & 2n-5 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & 1 & \cdots & -1 & -1 & 2n-5 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 1 & -1 & -1 & \cdots & 2n-5 \end{array}\right] \end{array}.$$

By using the block matrix, the Seidel Laplacian matrix of Γ_G can be derived as

$$SL(\Gamma_G) = \begin{bmatrix} 2I_{n-2} + J_{n-2} & -J_{(n-2)\times\frac{n}{2}} & -J_{(n-2)\times\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & (2n-4)I_{\frac{n}{2}} - J_{\frac{n}{2}} & (2I-J)_{\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & (2I-J)_{\frac{n}{2}} & (2n-4)I_{\frac{n}{2}} - J_{\frac{n}{2}} \end{bmatrix},$$

According to Theorem 3.4 with $r = 3$, $s = 2n - 5$, and $t = 1$, we get

$$P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda-2)^{n-3}(\lambda-2n+2)^{\frac{n}{2}}(\lambda-2n+6)^{\frac{n}{2}}.$$

It is obvious that $\lambda_1 = 2$ of multiplicity $n-3$, a single $\lambda_2 = 0$, $\lambda_3 = 2n-2$ of multiplicity $\frac{n}{2}$, $\lambda_4 = 2n-6$ of multiplicity $\frac{n}{2}$. So that the spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{(2n-2)^{\frac{n}{2}}, (2n-6)^{\frac{n}{2}}, (2)^{n-3}, (0)^1\right\}.$$

Taking the maximum absolute eigenvalues, then we derive the SL-spectral radius of Γ_G,

$$\rho_{SL}(\Gamma_G) = 2(n-1).$$

Using $Spec(\Gamma_G)$ we obtain the SL-energy for Γ_G as given below

$$E_{SL}(\Gamma_G) = \left(\frac{n}{2}\right)|2n-2| + \left(\frac{n}{2}\right)|2n-6| + (n-3)|2| + (1)|0| = 2(n^2-n-3).$$

□

Theorem 3.7. Let Γ_G be the commuting graph for D_{2n}, and $E_{SSL}(\Gamma_G)$ be the Seidel signless Laplacian energy of Γ_G. If $G = G_1$ or $G = G_2$, then

1. If $G = G_1$, then $E_{SSL}(\Gamma_G) = \begin{cases} (n-2)(n-1), & \text{if } n \text{ is odd} \\ (n-3)(n-2), & \text{if } n \text{ is even.} \end{cases}$
2. If $G = G_2$, then $E_{SSL}(\Gamma_G) = \begin{cases} n(n-1), & \text{if } n \text{ is odd} \\ 8, & \text{if } n = 4 \\ n(n-3), & \text{if } n \text{ is even} \end{cases}$

Proof.

1. By the same argument of Theorem 3.5 (1), the Seidel signless Laplacian matrix of Γ_G is an $(n-1)\times(n-1)$ matrix as follows:

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G) = \begin{bmatrix} 2-n & -1 & \cdots & -1 \\ -1 & 2-n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 2-n \end{bmatrix} = [(1-n)I_{n-1} - J_{n-1}].$$

Following Theorem 3.1 with $s = 2-n$, $t = -1$, and $m = n-1$, then the characteristic polynomial of $SSL(\Gamma_G)$

$$P_{SSL(\Gamma_G)}(\lambda) = (\lambda + 2(n-2))(\lambda + n - 3)^{n-2}.$$

Then the roots of $P_{SSL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = -2(n-2)$ and $\lambda_2 = 3-n$ with multiplicity $n-2$. Then the spectrum of Γ_G is

$$\sigma(\Gamma_G) = \begin{pmatrix} 3-n & -2(n-2) \\ n-2 & 1 \end{pmatrix}.$$

The Seidel signless Laplacian energy of Γ_G will be

$$E_{SSL}(\Gamma_G) = (n-2)|3-n| + (1)|-2(n-2)| = (n-2)(n-1).$$

Same idea for $G = G_1$ and n is even, from Theorem 3.5 (1), the Seidel signless Laplacian matrix of Γ_G is is an $(n-2)\times(n-2)$ matrix as follows:

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G) = \begin{bmatrix} 3-n & -1 & \cdots & -1 \\ -1 & 3-n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 3-n \end{bmatrix} = [(2-n)I_{n-2} - J_{n-2}].$$

Following Theorem 3.1 with $s = 3-n$, $t = -1$, and $m = n-2$, then the characteristic polynomial of $SSL(\Gamma_G)$

$$P_{SSL(\Gamma_G)}(\lambda) = (\lambda + 2(n-3))(\lambda + n - 4)^{n-3}.$$

Then the roots of $P_{SSL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = -2(n-3)$ and $\lambda_2 = 4-n$ with multiplicity $n-3$.

Then the spectrum of Γ_G is

$$\sigma(\Gamma_G) = \begin{pmatrix} 4-n & -2(n-3) \\ n-3 & 1 \end{pmatrix}.$$

The Seidel signless Laplacian energy of Γ_G will be

$$E_{SSL}(\Gamma_G) = (n-3)|4-n| + (1)|-2(n-3)| = (n-3)(n-2).$$

2. **When n is odd**. By the same argument of proofing part of Theorem 3.5 (2), then an $n \times n$ Seidel signless Laplacian matrix of Γ_G is

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G) = \begin{bmatrix} n-1 & 1 & \cdots & 1 \\ 1 & n-1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & n-1 \end{bmatrix}.$$

Following Theorem 3.1 with $s = n-1$, $t = 1$, and $m = n$, then the characteristic polynomial of $SSL(\Gamma_G)$

$$P_{SSL(\Gamma_G)}(\lambda) = (\lambda - 2(n-1))(\lambda - n + 2)^{n-1}.$$

Then the roots of $P_{SSL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = 2(n-1)$ and $\lambda_2 = n-2$ with multiplicity $n-1$. Thus, the spectrum of Γ_G is

$$Spec(\Gamma_G) = \{(2(n-1))^1, (n-2)^{n-1}\}.$$

The Seidel signless Laplacian energy of Γ_G will be

$$E_{SSL}(\Gamma_G) = (1)|2(n-1)| + (n-1)|n-2| = n(n-1).$$

When n is even. According to Theorem 3.5 (2), for $G = G_2$, then an $n \times n$ Seidel signless Laplacian matrix of Γ_G is

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G) = \begin{bmatrix} n-3 & 1 & \cdots & 1 & -1 & 1 & \cdots & 1 \\ 1 & n-3 & \cdots & 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & n-3 & 1 & 1 & \cdots & -1 \\ -1 & 1 & \cdots & 1 & n-3 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 1 & 1 & n-3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -1 & 1 & 1 & \cdots & n-3 \end{bmatrix}$$

$$= \begin{bmatrix} (n-4)I_{\frac{n}{2}} + J_{\frac{n}{2}} & -2I_{\frac{n}{2}} + J_{\frac{n}{2}} \\ -2I_{\frac{n}{2}} + J_{\frac{n}{2}} & (n-4)I_{\frac{n}{2}} + J_{\frac{n}{2}} \end{bmatrix}.$$

Following Theorem 3.2 with $s = n-3$, $t = 1$, then

$$P_{SSL(\Gamma_G)}(\lambda) = (\lambda - 2(n-3))(\lambda - n + 6)^{\frac{n}{2}-1}(\lambda + 2 - n)^{\frac{n}{2}}.$$

Then the roots of $P_{SSL(\Gamma_G)}(\lambda) = 0$ are a single $\lambda_1 = 2(n-3)$, $\lambda_2 = n-6$ with multiplicity $\frac{n}{2} - 1$, and $\lambda_3 = n-2$ with multiplicity $\frac{n}{2}$. Thus, the spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{(2(n-3))^1, (n-2)^{\frac{n}{2}}, (n-6)^{\frac{n}{2}-1}\right\},$$

and the Seidel signless Laplacian energy of Γ_G is

$$E_{SSL}(\Gamma_G) = (1)|2(n-3)| + \left(\frac{n}{2}\right)|n-2| + \left(\frac{n}{2} - 1\right)|n-6| = \begin{cases} 8, & \text{if } n = 4 \\ n(n-3), & \text{if } n > 4. \end{cases}$$

□

Theorem 3.8. Let Γ_G be the commuting graph for D_{2n}, where $G = G_1 \cup G_2 \subset D_{2n}$, then the SSL-energy for Γ_G is

1. for odd n, $E_{SSL}(\Gamma_G) = 2n^2 - 2n - 3 + \sqrt{(2n+1)^2 + 16(n-1)(n-3)}$,
2. for even n, $E_{SSL}(\Gamma_G) = 2(n^2 - 2n - 2 + \sqrt{5n^2 - 30n + 49})$.

Proof.

1. For $G_1 \cup G_2$ and odd n, from Theorem 3.6, the $(2n-1) \times (2n-1)$ Seidel signless Laplacian matrix of Γ_G is

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \begin{bmatrix} 2 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ -1 & 2 & \cdots & -1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 2(n-1) & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 2(n-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 2(n-1) \end{bmatrix} \end{array}.$$

By Theorem 3.3 with $s = 2$, $u = 2(n-1)$, dan $t = -1$, then

$$P_{SSL(\Gamma_G)}(\lambda) = (\lambda - 3)^{n-2}(\lambda^2 - (2n+1)\lambda - 4(n-1)(n-3))(\lambda - 2n + 3)^{n-1}.$$

Here we have four eigenvalues obtained from $P_{SSL(\Gamma_G)}(\lambda)$. They are $\lambda_1 = 3$ with multiplicity $n-2$, $\lambda_2 = 2n-3$ of multiplicity $n-1$ and $\lambda_{3,4} = \frac{2n+1}{2} \pm \frac{\sqrt{(2n+1)^2+16(n-1)(n-3)}}{2}$ of each multiplicity 1. Hence, the SSL –spectrum for Γ_G is as follows

$$Spec(\Gamma_G) = \left\{ \left(\frac{2n+1}{2} + \frac{\sqrt{(2n+1)^2+16(n-1)(n-3)}}{2} \right)^1, (2n-3)^{n-1}, (3)^{n-2}, \left(\frac{2n+1}{2} - \frac{\sqrt{(2n+1)^2+16(n-1)(n-3)}}{2} \right)^1 \right\}.$$

Now for $i = 1,2,3,4$, the maximum of absolute eigenvalues $|\lambda_i|$ is the SSL-spectral radius of Γ_G,

$$\rho_{SSL}(\Gamma_G) = \frac{2n+1}{2} + \frac{\sqrt{(2n+1)^2+16(n-1)(n-3)}}{2}.$$

By computing the eigenvalues from $Spec(\Gamma_G)$, then the SSL-energy for Γ_G is

$$\begin{aligned} E_{SSL}(\Gamma_G) &= (n-1)|2n-3| + (n-2)|3| + \left| \frac{2n+1}{2} \pm \frac{\sqrt{(2n+1)^2 + 16(n-1)(n-3)}}{2} \right| \\ &= 2n^2 - 2n - 3 + \sqrt{(2n+1)^2 + 16(n-1)(n-3)}. \end{aligned}$$

Suppose now n is even. Then a $(2n-1) \times (2n-1)$ Seidel signless Laplacian matrix of Γ_G is

$$SSL(\Gamma_G) = D(\Gamma_G) + S(\Gamma_G)$$

$$= \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^n \\ b \\ ab \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ a^{\frac{n}{2}+1}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} a & a^2 & \cdots & a^n & b & ab & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \cdots & a^{n-1}b \end{array} \\ \begin{bmatrix} 3 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ -1 & 3 & \cdots & -1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 3 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 2n-5 & 1 & \cdots & 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 2n-5 & \cdots & 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 2n-5 & 1 & 1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 & -1 & 1 & \cdots & 1 & 2n-5 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & -1 & \cdots & 1 & 1 & 2n-5 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & -1 & 1 & 1 & \cdots & 2n-5 \end{bmatrix} \end{array}.$$

By using the block matrix, the Seidel signless Laplacian matrix of Γ_G can be derived as

$$SSL(\Gamma_G) = \begin{bmatrix} 4I_{n-2} - J_{n-2} & J_{(n-2)\times\frac{n}{2}} & J_{(n-2)\times\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (2n-6)I_{\frac{n}{2}} + J_{\frac{n}{2}} & (J-2I)_{\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (J-2I)_{\frac{n}{2}} & (2n-6)I_{\frac{n}{2}} + J_{\frac{n}{2}} \end{bmatrix}.$$

By Theorem 3.4 with $r = 3$, $s = 2n-5$, and $t = -1$, we derive

$$P_{SSL(\Gamma_G)}(\lambda) = (\lambda - 4)^{n-3}(\lambda - 2n + 8)^{\frac{n}{2}-1}(\lambda - 2n + 4)^{\frac{n}{2}}(\lambda^2 - 2(n-1)\lambda - 4(n^2 - 7n + 12).$$

It is obvious that $\lambda_1 = 4$ of multiplicity $n-3$, $\lambda_2 = 2n-8$ of multiplicity $\frac{n}{2} - 1$, $\lambda_3 = 2n-4$ of multiplicity $\frac{n}{2}$ and the other two eigenvalues are $\lambda_{4,5} = n - 1 \pm \sqrt{5n^2 - 30n + 49}$. So that the spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{ \left(n - 1 + \sqrt{5n^2 - 30n + 49}\right)^1, (2n-4)^{\frac{n}{2}}, (2n-8)^{\frac{n}{2}-1}, (4)^{n-3}, \left(n - 1 - \sqrt{5n^2 - 30n + 49}\right)^1 \right\}.$$

Taking the maximum absolute eigenvalues, then we derive the SSL-spectral radius of Γ_G,

$$\rho_{SSL}(\Gamma_G) = n - 1 + \sqrt{5n^2 - 30n + 49}.$$

Using $Spec(\Gamma_G)$ we obtain the SSL-energy for Γ_G as given below

$$\begin{aligned} E_{SSL}(\Gamma_G) &= \left(\frac{n}{2} - 1\right)|2n - 8| + \left(\frac{n}{2}\right)|2n - 4| + (n - 3)|4| + \left|n - 1 \pm \sqrt{5n^2 - 30n + 49}\right| \\ &= 2\left(n^2 - 2n - 2 + \sqrt{5n^2 - 30n + 49}\right). \end{aligned}$$

□

Discussion

As a result of Theorem 3.6 and 3.8, we obtain the classification of the Seidel Laplacian and Seidel signless Laplacian energies of Γ_G for D_{2n}.

Corollary 4.1. Let $G = G_1 \cup G_2 \subset D_{2n}$, Γ_G is hyperenergetic corresponding to Seidel Laplacian and Seidel signless Laplacian matrices.

Corollary 4.2. Let Γ_G be the commuting graph for D_{2n}, where $G = G_1 \cup G_2 \subset D_{2n}$, then the Seidel Laplacian energy for Γ_G is always an even integer.

Corollary 4.3. Let Γ_G be the commuting graph for D_{2n}, where $G = G_1 \cup G_2 \subset D_{2n}$, then the Seidel signless Laplacian energy for Γ_G is never an odd integer.

These facts comply with the well-known results from Bapat & Pati (2004) and Pirzada & Gutman (2008). Furthermore, the relationship between SL and SSL-energies are presented in the next two corollaries.

Corollary 4.4. Let Γ_G be the commuting graph for D_{2n}, where $G = G_1$ or $G = G_2$, then

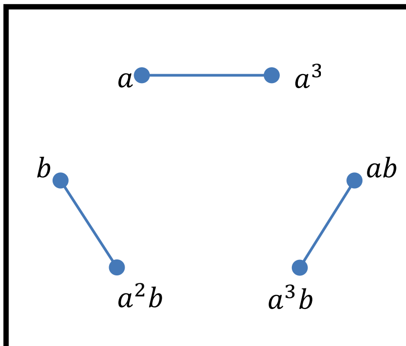
$$E_{SL}(\Gamma_G) \begin{cases} < E_{SSL}(\Gamma_G), & \text{if } n = 4 \\ = E_{SSL}(\Gamma_G), & \text{otherwise.} \end{cases}$$

Corollary 4.5. Let Γ_G be the commuting graph for D_{2n}, where $G = G_1 \cup G_2 \subset D_{2n}$, then

$$E_{SL}(\Gamma_G) \begin{cases} = E_{SSL}(\Gamma_G), & \text{if } n = 3 \text{ or } 4 \\ \leq E_{SSL}(\Gamma_G), & \text{otherwise.} \end{cases}$$

The following example is an illustration of Theorem 3.6 and 3.8 for $n = 8$.

Example 1. The Seidel Laplacian matrix of Γ_G is as in Figure 1, where $G = G_1 \cup G_2 \subset D_8$, $G_1 = \{a, a^3\}$ and $G_2 = \{b, ab, a^2b, a^3b\}$.



$$SL(\Gamma_G) = \begin{bmatrix} 3 & 1 & -1 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 1 & -1 \\ -1 & -1 & -1 & 3 & -1 & 1 \\ -1 & -1 & 1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 1 & -1 & 3 \end{bmatrix}$$

$$SSL(\Gamma_G) = \begin{bmatrix} 3 & -1 & 1 & 1 & 1 & 1 \\ -1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 & 1 & -1 \\ 1 & 1 & -1 & 1 & 3 & 1 \\ 1 & 1 & 1 & -1 & 1 & 3 \end{bmatrix}$$

Figure 1. [Commuting graph for $G = G_1 \cup G_2 \subset D_8$]

In this case, $P_{SL(\Gamma_G)}(\lambda) = \lambda(\lambda - 2)^3(\lambda - 6)^2$ implies the eigenvalues of $SL(\Gamma_G)$ are $\lambda = 2$ with multiplicity (3), $\lambda = 6$ with multiplicity (2), and a single $\lambda = 0$. Hence, $E_{SL}(\Gamma_G) = (3)|2| + (2)|6| + (1)|0| = 18$, conforming Theorem 3.6 for even n. Meanwhile, $P_{SSL(\Gamma_G)}(\lambda) = \lambda(\lambda - 4)^3(\lambda^2 - 6\lambda)$ implies the eigenvalues of $SSL(\Gamma_G)$ are $\lambda = 4$ with multiplicity (3), $\lambda = 0$ with multiplicity (2), a single $\lambda = 6$, Hence, $E_{SSL}(\Gamma_G) = (3)|4| + (2)|0| + (1)|6| = 18$, conforming Theorem 3.8 for even n. We conclude in this example that $E_{SL}(\Gamma_G) = E_{SSL}(\Gamma_G)$.

Conclusion

We presented the spectrum and spectral radius of Γ_G for dihedral groups, D_{2n}, where $n \geq 3$, which are linked to the Seidel Laplacian and Seidel signless Laplacian matrices. Then, the Seidel Laplacian and Seidel signless Laplacian energies of Γ_G is presented for each of the following cases: G_1, G_2 or $G_1 \cup G_2$. Our research has demonstrated that the Seidel Laplacian and Seidel signless Laplacian energies of Γ_G, in line with previous publications, never takes the form of an odd integer. Those energies are equal whenever $n = 3$ or 4 and otherwise, the Seidel signless Laplacian energy is never less than the Seidel Laplacian energy of Γ_G. Moreover, we emphasize that Γ_G possesses hyperenergy.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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