DIAGONALLY IMPLICIT MULTISTEP BLOCK METHOD FOR SOLVING DELAY VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

By

NUR AUNI BINTI BAHARUM

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To the ones who are my most significant source of strength and hope,
To the ones who choose to believe me in this journey,
To the ones who put their trust in me,
...
My parents:
Baharum bin Sulaiman,
Mek Yah binti Ya.
...
My siblings:
Asrul Aiman bin Baharum,
Hafidzul Aiman bin Baharum,
Nur Alya binti Baharum,
Haziqah binti Isa,
Aina Aqilah binti Asrul Aiman,
Muhammad Haqq bin Asrul Aiman.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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May 2023

Chairman : Professor Zanariah Abdul Majid, PhD
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In this study, two points diagonally implicit multistep block (2DIMB) methods are constructed for the numerical solution of the first and second order delay Volterra integro-differential equation (DVIDE). The second order of DVIDE is solved directly without reducing the problem in the system of the first order of DVIDE. Two distinct types of DVIDE are solved, namely unbounded and bounded time lag cases. Furthermore, the constant and pantograph delay types indicate that the delay conditions for DVIDE are also considered in this study. The strategy of the constant step size is implemented for finding the numerical solution to DVIDE.

When finding the approximate solution to DVIDE, three components must be considered: the initial value problem of DVIDE, the delay solution, and the integral part. The 2DIMB method is formulated for the numerical solution of initial value problem of DVIDE and computed two solutions simultaneously in block form. This method is built on a predictor-corrector formula.

The previously calculated solutions are used to obtain the delay solution for the constant delay type. Meanwhile, Lagrange interpolation polynomial is implemented to approximate the delay solution for the pantograph delay type. Since an integral part of DVIDE cannot be solved explicitly and analytically, the idea of approximating the solution is discussed. The appropriate order of the numerical integration method is chosen to approximate the solution of the integral part of DVIDE, which include trapezoidal rule, Simpson's rule, and Boole's rule.

Analysis on order, error constants, consistency, zero-stability, and convergence of the proposed method are given in this study. Moreover, the stability region is discussed based on the stability polynomial of the 2DIMB method paired with the appropriate numerical integration method. All the computational procedures were undertaken using the C programming language in a CODE::BLOCKS platform.

Numerical results showed that where the proposed methods are reliable and suitable for solving the unbounded and bounded time lag of the DVIDE for the constant and pantograph delay types. Three advantages in terms of the total steps taken, function evaluations and the execution time taken by these methods have been identified when comparing the numerical results with the Runge-Kutta and Adam-Bashforth-Moulton methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH BLOK MULTILANGKAH TERSIRAT PEPENJURU DIGUNAKAN UNTUK MENYELESAIKAN PERSAMAAN PERBEZAAN LENGAH KAMIRAN VOLTERRA

Oleh

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Dalam kajian ini, kaedah blok multilangkah dua titik tersirat pepenjuru (BM2TTP) dibangunkan untuk penyelesaian berangka bagi peringkat pertama dan kedua persamaan perbezaan lengah kamiran Volterra (PPLKV). Peringkat kedua PPLKV perlu diselesaikan secara langsung tanpa menurunkan kepada sistem peringkat pertama PPLKV. Dua jenis PPLKV yang berbeza diketengahkan untuk diselesaikan: kes lengah masa yang tidak terbatas dan terbatas. Seterusnya, jenis lengah pemalar dan pantograf yang menunjukkan keadaan lengah untuk PPLKV juga dipertimbangkan dalam kajian ini. Strategi ukuran langkah yang tetap dilaksanakan untuk mencari penyelesaian berangka bagi PPLKV.

Tiga komponen mesti dipertimbangkan semasa mencari penyelesaian berangka bagi PPLKV: masalah nilai awal PPLKV, penyelesaian lengah dan bahagian kamiran. Kaedah BM2TTP dirumuskan untuk mencari penyelesaian berangka bagi masalah nilai awal PPLKV dan mengira dua penyelesaikan secara serentak dalm bentuk blok. Kaedah ini dibina berdasarkan formula peramal-pembetulan.

Penyelesaikan yang dikira sebelum ini digunakan untuk menyelesaikan masalah lengah bagi jenis lengah pemalar. Sementara itu, polinomial interpolasi Lagrange diimplementasikan untuk mengira masalah lengah bagi jenis lengah pantograf. Disebabkan bahagian kamiran PPLKV tidak dapat diselesaikan secara jelas dan analitik, satu idea anggaran penyelesaian dibincangkan. Formula kamiran berangka yang mempunyai urutan yang sesuai dipilih bagi mencari penyelesaian untuk bahagian kamiran PPLKV yang merangkumi petua trapezium, petua Simpson dan petua Boole.

Analisis yang merangkumi ciri-ciri peringkat, pemalar ralat, konsistensi, kestabilan-sifar dan penumpuan kaedah yang dicadangkan dikaji dalam kajian ini. Tambahan pula, rantau kestabilan dibincangkan berdasarkan kestabilan polinomial untuk kaedah BM2TTP yang digandingkan dengan kaedah pengamiran berangka yang sesuai. Semua prosedur pengiraan dilakukan dengan menggunakan bahasa pengaturcaraan C dalam perisian CODE::BLOCKS.

Keputusan berangka menunjukkan penemuan penting di mana kaedah yang dicadangkan boleh dipercayai dan sesuai untuk menyelesaikan masalah PPLKV bagi kes penundaan masa yang tidak terbatas dan terbatas untuk jenis penundaan pemalar dan pantograf. Tiga kelebihan dari segi jumlah bilangan langkah, penilaian fungsi dan masa pelaksaaan yang diambil oleh kaedah ini telah dikenal pasti apabila membandingkan keputusan berangka dengan kaedah Runge-Kutta dan Adam-Bashforth-Moulton.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

τ	Delay term
$(x-\tau)$	Delay argument
$y(x-\tau)$	Delay solution
y_d	y approximate for the delay solution
DDE	Delay differential equations
IVP	Initial value problems
LMM	Linear multistep method
ODE	Ordinary differential equations
VIDE	Volterra integro-differential equations
DVIDE	Delay Volterra integro-differential equations
2DIMB	Two points diagonally implicit multistep block method
2DIMB3	Two points diagonally implicit multistep block method of order three
2DIMB4	Two points diagonally implicit multistep block method of order four
2DIMB5	Two points diagonally implicit multistep block method of order five
2O2DDI	Two points direct diagonally implicit multistep block method

CHAPTER 1

INTRODUCTION AND PRELIMINARY MATHEMATICAL CONCEPT

1.1 Background

Numerous real-life phenomena in physics, engineering, biology, medicine, and economics can be modelled using an initial value problems (IVP) for the ordinary differential equations (ODE) of the type;

$$\begin{aligned} y'(x) &= F\big(x, y(x)\big), \qquad x \geq x_0, \\ y(x_0) &= y_0, \end{aligned} \tag{1.1.1}$$

where y' is the derivative of unknown function x and F is a continuous function. The function $y(x)$, referred to as the state variable, reflects an evolving physical quantity over time, x. However, it is occasionally essential to make changes to the right-hand side of the equation (1.1.1) to make the model more consistent with the real-life phenomena.

Delay differential equations (DDE) have gained significant interest in scientific areas over the decades. Delays (hereditary, memories, retarded arguments post-actions, dead times, or time delays) are innate to many physical and engineering systems. It has developed into a potent instrument for probing the intricacies of real-life problems, including infectious illness, population dynamics, neuronal networks, and even economics and finance. DDE is described in mathematics as a differential equation in which the derivatives of certain unknown functions at present depend on the values of the functions at the previous time. The other names for this equation include a time-delay system, deviating argument equations, differential-difference equations, and an ODE with a time lag. The standard form of DDE is denoted as;

$$\begin{aligned} y'(x) &= F\big(x, y(x), y(x-\tau)\big), \\ y(x) &= \phi(x), \end{aligned} \tag{1.1.2}$$

where $\phi(x)$ is the arbitrary initial function. The $\tau = \tau\big(x, y(x)\big)$ represents a delay term, while $(x-\tau)$ indicates a delay argument and $y(x-\tau)$ denotes the delay solution.

In mathematics, ODE and DDE are considered to share a similarity in that they both seek unique solutions, and both arise from the study of precisely solvable physical phenomena. Despite the apparent similarities, there are some essential differences between ODE and DDE, emphasized in Table 1.1.

Table 1.1: The distinctions between first order ODE and DDE.

ODE	DDE
Standard form: $y'(x) = F\big(x, y(x)\big)$, an equation without the presence of delays	**Standard form:** $y'(x) = F\big(x, y(x), y(x-\tau)\big)$, an equation with the presence of delays
Initial value: at point $y(a) = y_0$, to determine a unique solution $y(x)$.	**Initial value and initial function:** at point $y(a) = y_0$, to determine a unique solution $y(x)$. $y(x) = \phi(x)$, to determine a unique solution for $y(x-\tau)$.
Solution: The unique solution of $y(x)$ is evaluated at a specific time of x.	**Solution:** The unique solutions of $y(x)$ are evaluated at both particular times of x and the previous time for the location of the delay.

Meanwhile, Volterra began exploring integral equations in 1844 and took the study seriously in 1896 (Wazwaz, 2011). Volterra investigated hereditary influences while researching a population growth model. The research produced a specific topic in which differential and integral operators coexisted in the same equation. This novel type of equation was termed Volterra integro-differential equations (VIDE) as,

$$y'(x) = f\big(x, y(x)\big) + \int_0^x K(x,u)y(u)\,du, \tag{1.1.3}$$

where $K(x,u)$ is a known function of two variables x and u, called the kernel function. VIDE emerged in many scientific and engineering applications, including electrical circuit analysis, viscoelastic and heat transfer.

VIDE is classified into two categories based on the homogeneity and linearity concepts. The concepts of homogeneity and linearity considerably impact the solution structures. The equation is homogeneous if the function $f(x)$ in VIDE is identically zero, otherwise inhomogeneous. When the power of $y(u)$ within the integral component is one, VIDE is classified as linear. However, a nonlinear function of VIDE arises when the power of the unknown function $y(u)$ in the integral part exceeds one or when it contains nonlinear functions of $y(u)$ such as $e^{(y)}$, $\sin(y)$, $\cosh(y)$ and $\ln(1+y)$.

Nonetheless, this study explores a unique equation where the DDE and VIDE appear in the same equation. Volterra introduced this unique equation when he studied some delay models in his work on population dynamics in the early 1920s and 1930s. While it is widely acknowledged that delays play a crucial role in population dynamics (and biology in general), VIDE models with delays have been developed and examined increasingly frequently in recent years, as indicated by the growing presence of literature on the subject (Cushing, 1977). The equation that unifies the theories of DDE and VIDE is referred to as the delay Volterra integro-differential equation (DVIDE), or VIDE with deviating argument.

Incorporating the delay element in the integro-differential equation to model real-life phenomena has increased dramatically during the last few decades. Such models have a wide variety concerning the integro-differential equation and how the delay element appears in the underlying equation. DVIDE encompasses a broad spectrum of fields, from biology to control problems, materials science, and economics (Kolmanovskii and Myshkis, 2012; Baker, 2000; Baker et al., 1999). The delay term makes the DVIDE too complex to hope for analytical solutions. The analytical solution is sometimes impracticable and needs to be improved in giving the required solution. Therefore, reliable numerical schemes are needed to obtain solutions to such equations and come to intrigue researchers in numerical computation and analysis.

1.2 Delay Volterra Integro-differential Equation

Consider the delay Volterra integro-differential equation,

$$y'(x) = F\left(x, y(x), y(x-\tau), \int_{a(x)}^{x} K(x,u)y(u)y(u-\tau)\,du\right). \tag{1.2.1}$$

The classification of equation (1.2.1) can be naturally expanded with numerous delay types to DVIDE. This study considers the delay Volterra integro-differential equation with definite integral. The general form of the first order DVIDE with definite integral is considered as,

$$\begin{aligned} y'(x) &= F\big(x, y(x), y(x-\tau), z(x)\big), \qquad && a \le x \le b, \\ \text{where} \quad z(x) &= \int_{a(x)}^{x} K(x,u)y(u)y(u-\tau)\,du, && \\ \text{with} \quad \phi(x) &= y(x), && x \in [-\tau, x_0]. \end{aligned} \tag{1.2.2}$$

Consequently, two cases of the DVIDE with definite integral are introduced, which depend on the value of the delay argument at the limit of integration. The general form of delay Volterra integro-differential equation is

1. Unbounded time-lag
   The delay argument does not occur at the integration limit, i.e., $a(x) = 0$ and has fixed values at the limit of integration, (Rihan et al., 2009),

$$y'(x) = F\left(x, y(x), y(x-\tau), \int_{0}^{x} K(x,u)y(u)y(u-\tau)\,du\right). \tag{1.2.3}$$

2. Bounded time-lag
   The delay argument does occur at the integration limit, i.e., $a(x) = x - \tau$ and has unfixed values at the limit of integration, (Rihan et al., 2009),

$$y'(x) = F\left(x, y(x), y(x-\tau), \int_{x-\tau}^{x} K(x,u)y(u)y(u-\tau)\,du\right). \qquad (1.2.4)$$

In this case, the delay or lag, τ is measurable as a physical quantity that is a scalar in a function. It is always a continuous function and non-negative values. The arbitrary initial function, $\phi(x)$ is understood to be defined in $[\rho, x_0]$, where,

$$\rho = \min_{1\leq i\leq n}\left\{\min_{x\geq x_0}(x-\tau)\right\}.$$

The delay terms for DVIDE are estimated first before approximating the unique solution $y(x)$. The delay argument, $(x-\tau)$ lies within the interval $[x_0, X]$, where if $(x-\tau) \leq x_0$, then the initial function $y(x-\tau) = \phi(x-\tau)$ need to be applied. Elseways, when $(x-\tau) > x_0$, an interpolation polynomial must be applied in finding the solution of the delay argument. These are four conditions by which the delay can be represented;

1) Constant delay, where $\tau = \mathbb{R}$.
   Example:

$$\begin{aligned} y'(x) &= 1 - \frac{x^4}{3} + \int_0^x xuy(u-1)\,du, \\ y(0) &= 1, \qquad 0 \leq x \leq 1, \end{aligned}$$

   where $x - \tau(x) = x - 1$, thus $\tau(x) = 1 \in \mathbb{R}$.

2) Proportional delay (Pantograph delay), where τ is a function of x but the coefficient of $x \in [0,1]$.
   Example:

$$\begin{aligned} y'(x) &= y^2\left(\frac{x}{2}\right) - e^x + 1 + \int_0^x y^2\left(\frac{u}{2}\right)du, \qquad x \in [0,1], \\ \phi(x) &= e^x, \qquad x \leq 0, \end{aligned}$$

   where $x - \tau(x) = \frac{x}{2}$, thus $\tau(x) = x - \frac{x}{2} = \frac{x}{2}$.

3) Time-dependent delay, where τ is a function of x.
   Example:

$$\begin{aligned} y'(x) &= xe^x - e^{-x} + \int_{x-e^x}^{x} x\exp(2u)\,y(u)\,y'(u)\,du, \\ \phi(x) &= e^{-x}, \qquad x \leq 0, \end{aligned}$$

where $x-\tau(x)=x-e^x$, thus $\tau(x)=e^x$.

4) State-dependent delay, where τ is a functions of both x and $y(x)$.
   Example:

$$y'(x)=\int_0^{y(x)} y(u)\,du, \qquad x\geq 2,$$
$$\phi(x)=1, \qquad x\leq 2,$$

   where $x-\tau(x)=y(x)$, thus $\tau(x)=x-y(x)$.

The first and second order DVIDE with constant and pantograph delay conditions have been studied in this study. A standard form of the second order DVIDE is considered as follows,

$$y''(x)=F\left(x,y(x),y(x-\tau),y'(x),y'(x-\tau),\int_0^x K(x,u)\,y(u)\,y'(u)\,y'(u-\tau)\,du\right),$$
$$\text{where } x\in[x_0,X],$$
$$y(x)=\phi(x), \quad x\leq x_0, \qquad y'(x)=\phi'(x), \quad x\leq x_0. \tag{1.2.5}$$

Numerical solution for solving the second order DVIDE directly using the 2O2DDI is described. The provided initial function or Lagrange interpolation polynomial estimates the delay terms.

1.3 Linear Multistep Method

Typically, numerical methods for solving the IVP problem of ordinary differential equations fall into one or two broad categories: one-step methods (e.g., Euler method or Runge-Kutta method) or linear multistep methods (e.g., Adam methods). The one-step method is a self-starting method that approximates the solution at x_{n+1} using information from one of the previous points, x_n. The starting point for the numerical solution of this method depends only on the initial condition, and there is no iterative procedure involved in obtaining an approximation of the solution.

Several researchers, including Gragg and Stetter (1964), Butcher (1965), Gear (1965) and Lambert (1973), proposed the modified linear multistep method, which was demonstrated to be capable of overcoming the Dahquist barrier theorem. Incorporating off-step points into the derivation process yielded this method. The general linear multistep method, (LMM) can be defined as,

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h\sum_{j=0}^{k}\beta_j f_{n+j}, \tag{1.3.1}$$

where the coefficient $\alpha_0, \ldots, \alpha_j$ and $\beta_0, \ldots, \beta_j$ are real constants and assume that $\alpha_k \neq 0$ and that α_0 and β_0 are not both equal to zero in order to avoid degenerate cases.

This approach is the multistep method because it depends on the approximation at multiple previous mesh points to determine the approximation at the subsequent point. This method's numerical solution depends on the initial values and necessitates an iterative process to reach a sufficiently comparative value. This method is also referred to as a predictor-corrector method. The method is said to be explicit (predictor) when $\beta_k = 0$ and if $\beta_k \neq 0$, then the method is called implicit (corrector).

Definition 1.1 (Linear difference operator)
The linear difference operator L associated with the linear multistep method (1.3.1) *is defined by,*

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x+jh) - h\beta_j y'(x+jh) \right], \tag{1.3.2}$$

where $y(x)$ is an arbitrary function and continuously differentiable on $[a,b]$.
Source: Lambert (1973).

Expanding the function $y(x+jh)$ and its derivative $y'(x+jh)$ as Taylor series about x;

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots + \frac{h^n}{n!}y^{(n)}(x). \tag{1.3.3}$$

Hence, collecting terms in (1.3.2) gives,

$$L[y(x):h] = C_0 y(x) + C_1 h y^{(1)}(x) + \ldots + C_p h^p y^{(p)}(x), \tag{1.3.4}$$

where the C_0, C_1, C_2, ..., C_p are constants. The Taylor series expansion will be truncated based on the order of the method and substituted in equation (1.3.4) to determine the proposed multistep method.

Definition 1.2 (Order)
The linear multistep method (1.3.1) *is said to be of order p if,*

$$C_0 = C_1 = \ldots = C_{p+D-1} = 0,$$

and $C_{p+D} \neq 0$ is called as error constant of the method and D is the order of the equation.
Source: Lambert (1973) and Fatunla (1995).

The general formulae for constants C_p for the first order problem is developed as follows;

$$C_p = \sum_{j=0}^{k} \left[\frac{j^p \alpha_j}{p!} - \frac{j^{(p-1)} \beta_j}{(p-1)!} \right], \qquad p = 0, 1, 2, \ldots. \tag{1.3.5}$$

While for the second order problem, the calculation of the error constant is determined as follows,

$$C_p = \sum_{j=0}^{k} \left[\frac{j^p \alpha_j}{p!} - \frac{j^{(p-1)} \beta_j}{(p-1)!} - \frac{j^{(p-2)} \gamma_j}{(p-2)!} \right], \qquad p = 0, 1, 2, \ldots. \tag{1.3.6}$$

Definition 1.3 (Consistency)
The linear multistep method (1.3.1) *is said to be consistent if it has order $p \geq 1$ and the method is consistent if and only if,*

$$\sum_{j=0}^{k} \alpha_j = 0, \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j.$$

Source: Lambert (1973).

Associated with the general linear multistep method (1.3.1) given is a polynomial, the characteristic polynomial of the method is called as a first characteristic polynomial,

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \qquad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j. \tag{1.3.7}$$

From (1.3), the linear multistep method is consistent if and only if,

$$\rho(1) = 0, \qquad \rho'(1) = \sigma(1). \tag{1.3.8}$$

Definition 1.4 (Zero stability for linear multistep method)
The linear multistep method (1.3.1) *is said to be zero-stable if no root of the first characteristic polynomial* (1.3.7) *has modulus greater than one.*
Source: Lambert (1973).

Definition 1.5 (Zero stability for block method)
The block method is zero stable provided the roots R_m, $m = 1(1)k$ of the first characteristic polynomial $\rho(R)$ specified as,

$$\rho(R) = \det \left[\sum_{n=0}^{k} A^{(n)} R^{k-n} \right] = 0, \qquad A^{(0)} = -I, \tag{1.3.9}$$

satisfied with $| R_m | \leq 1$ and those roots with $|R| = 1$, the multiplicity must not exceed two.
Source: Fatunla (1991).

Theorem 1.1 (Convergence of linear multistep method)
The linear multistep method is said to be convergent if and only if the method are consistent and zero-stable.
Source: Lambert (1973).

Definition 1.6 (Convergence of the method)

$$\lim_{n\to\infty} y(x_i) = Y(x_i),$$

is the convergence condition for the approximate where $y(x_i)$ is the approximate solution and $Y(x_i)$ is the exact solution.
Source: Brunner and Lambert (1974).

1.4 Lipschitz Condition

Let,

$$\begin{aligned} R_1 &= (x,y,y_d,z) : 0 \le x \le b, \quad |y| < \infty, \quad |y_d| < \infty, \quad |z| < \infty, \\ R_2 &= (x,u,y,y_d) : 0 \le x \le b, \quad |y| < \infty, \quad |y_d| < \infty. \end{aligned} \tag{1.4.1}$$

Equation (1.2.2) defines points in R_1 and R_2 and the following conditions is considered as,

1. F and K are uniformly continuous in each variable.
2. For the F function and all (x,y,y_d,z) and $(x,\tilde{y},\tilde{y_d},z)$ in R_1,

$$\begin{aligned} \left|F(x,y,y_d,z) - F(x,\tilde{y},y_d,z)\right| &\le L_1|y-\tilde{y}|, \\ \left|F(x,y,y_d,z) - F(x,y,\tilde{y_d},z)\right| &\le L_2|y_d-\tilde{y_d}|, \\ \left|F(x,y,y_d,z) - F(x,y,y_d,\tilde{z})\right| &\le L_3|z-\tilde{z}|. \end{aligned} \tag{1.4.2}$$

3. For the K function and all (x,u,y,y_d) and $(x,u,\tilde{y},\tilde{y_d})$ in R_2,

$$\begin{aligned} \left|K(x,u,y,y_d) - K(x,u,\tilde{y},y_d)\right| &\le L_4|y-\tilde{y}|, \\ \left|K(x,u,y,y_d) - K(x,u,y,\tilde{y_d})\right| &\le L_5|y_d-\tilde{y_d}|. \end{aligned} \tag{1.4.3}$$

4. F_y, F_{y_d}, F_z, K_y and K_{y_d} functions are continuous and satisfy by the following condition;

$$\begin{aligned} &F_y(x,y,y_d,z) \ge 0, \quad F_{y_d}(x,y,y_d,z) \ge 0, \quad F_z(x,y,y_d,z) \ge 0, \\ &K_y(x,y,y_d,z) \ge 0, \quad K_{y_d}(x,y,y_d,z) \ge 0, \end{aligned}$$

for all (x, y, y_d, z) in R_1 and (x, u, y, y_d) in R_2.

It is well-known that, under these conditions, equation (1.2.2) possesses a unique solution $y(x) \in C^1[0, b]$.

1.5 Diagonally Implicit Multistep Method

There are two types of implicit multistep methods: the fully implicit multistep method and the diagonally implicit multistep method. Since the fully implicit multistep method requires extra information or points to approximate the solution, thus the problem of evaluating the steps becomes much more complicated and potentially more costly.

Diagonally implicit multistep methods were introduced by Butcher (1993). A method must also have a diagonally implicit structure to be the diagonally implicit multistep method. This means the $s \times s$ matrix A has the form;

$$A = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda & 0 & \cdots & 0 \\ a_{31} & a_{32} & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & a_{s3} & \cdots & \lambda \end{bmatrix},$$

where $\lambda \geq 0$. This restriction on this coefficient matrix is based on the fact that the steps can be computed sequentially or in parallel if the lower triangular component of A is zero. This will lead to a considerable saving over a method in which A has a general implicit structure, (Butcher, 2008).

1.6 Preliminary Mathematical Concept

Consider the general linear multistep method for DVIDE as;

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i F\left(x_{n+i}, y_{n+i}, y_{n-m+i}, z_{n+i}\right), \tag{1.6.1}$$

where the class of appropriate quadrature formulae is

$$z_n = h \sum_{i=0}^{n} \omega_{ni} K(x_n, x_i, y_i). \tag{1.6.2}$$

Definition 1.7 (Order of the method)
The difference operators L and M associated with the combination method are given as,

$$L[y(x_n);h] = \sum_{i=0}^{k} \Big(\alpha_i y(x_{n+i}) - h\beta_i y'(x_{n+i})\Big), \qquad n = 0,1,\ldots,N-k,$$

and

$$M[y(x_n);h] = \sum_{i=0}^{k} \Big(\alpha_i y(x_{n+i}) - h\beta_i F\left(x_{n+i}, y(x_{n+i}), y(x_{n-m+i}), z(x_{n+i})\right)\Big).$$

The order of L is defined as the order of (ρ,σ).
Source: Brunner and Lambert (1974).

Definition 1.8 (Order of the method)
Let L be in order p, let quadrature rule have order q. Then we define the order r of the combination method by $r = min(p,q)$.
Source: Brunner and Lambert (1974).

1.7 Motivation

Delay Volterra integro-differential equation has a wide range of applications, and as a result, finding the solution to DVIDE has received considerable attention. Numerical methods have become more prevalent as computer technology has progressed. However, numerous numerical methods exist and must be more appropriate for locating DVIDE solutions.

The development of the multistep block method for solving numerous initial value problems or differential equations is widely recognized. Nevertheless, the implementation and performance of the multistep block method in DVIDE have yet to be thoroughly studied.

Consequently, the motivation of this thesis is to generate the two points diagonally implicit multistep block method with the same order between the first and second points of the method. The development of the numerical scheme based on a diagonal formula demonstrates that the diagonally implicit multistep block method is significantly cheaper in computational effort and competes favorably with the existing methods. The proposed method is practical to retain the high accuracy of the computed results.

1.8 Problem Statement

The analytical solution to the delay Volterra integro-differential equation is excessively complicated. Qualitative results require a reliable numerical method (Yüzbaşı and Karaçayır, 2018) since DVIDE is challenging to solve analytically. Most numerical methods for solving delay Volterra integro-differential equations, such as the Runge-Kutta method, Galerkin's method and the spline collocation method, produce only one new approximation value at each step. Also, the multistep block method for solving the delay Volterra integro-differential equation has yet to be studied in detail. Hence, by taking this golden opportunity to investigate DVIDE with the diagonally implicit multistep block method. The proposed method is also known as an implicit method, and theoretically, the implicit method is more accurate than the explicit method.

1.9 Objectives of Study

The objectives can be specified as follows:

1. To derive two points diagonally implicit multistep block method (2DIMB) using Taylor series polynomial to solve the first order delay Volterra integro-differential equation.

2. To formulate two points direct diagonally implicit multistep block method (2O2DDI) for directly solving the second order delay Volterra integro-differential equation using Lagrange interpolation polynomial.

3. To conduct detailed analysis of the method's properties, including order, stability, consistency, and convergence.

4. To develop the algorithm and C programming language for the 2DIMB and 2O2DDI methods to solve the first and second order DVIDE with constant and pantograph delays in unbounded and bounded time lags.

1.10 Scope of the Study

The delay Volterra integro-differential equation with retarded type is the subject of this thesis. The following two cases of DVIDE are discussed in detail: unbounded time lag and bounded time lag. Furthermore, the constant and proportional delay conditions are the objective of the study.

This thesis provides in-depth details on developing new algorithms to solve the first and second order DVIDE numerically. Three orders of the 2DIMB method, i.e., third, fourth and fifth orders, have been derived and implemented to solve the first order DVIDE. Meanwhile, the second order DVIDE has been solved directly using the fourth order 2O2DDI. This thesis investigated the analysis of these methods, including order, consistency, zero-stability, convergence, and stability. The constant step size strategy is adapted while developing the C language algorithm for DVIDE.

1.11 Outline of the Thesis

This thesis is comprised of eight chapters and is structured as follows.

Chapter 1 introduces the delay Volterra integro-differential equation briefly. DVIDE is discussed in detail in terms of its cases and types. This thesis also discusses the study's motivation, problem statement, objectives, and scope. Furthermore, this chapter emphasizes the relevant mathematical concept of DVIDE and discusses the fundamental definitions and theorems underlying the numerical method. The numerical formulae are derived using the theories of Taylor series polynomials.

Chapter 2 briefly reviews the prior works as the chronological studies that led to the research roadmap and focus on the three parts of the literature reviews, such as the block method, DVIDE of the unbounded time-lag and DVIDE of the bounded time-lag.

Chapter 3 concentrates on deriving the three, four, and five-order of two points diagonally implicit multistep block method for the constant step size strategy. The analysis of these multistep block methods has been discussed in detail, including the proposed method's order, consistency, zero stability and convergence analysis. The Lagrange interpolation polynomial introduced in this chapter will be used to determine the delay solution of the problem. Meanwhile, the numerical integration method is introduced to tackle the integral part of DVIDE.

Chapter 4 deals with the numerical solution for the third, fourth and fifth order methods of 2DIMB to solve the delay Volterra integro-differential equation with unbounded time-lag of constant delay condition. Considering the appropriate numerical integration method for handling the solution of the integral part of DVIDE will be emphasized. The algorithm that has been designed will be implemented in the constant step size strategy. Therefore, this chapter discusses the numerical results of the tested problems and their comparison with other methods.

Chapter 5 focuses on the bounded time-lag of DVIDE with constant delay conditions. Moreover, this chapter discusses the numerical treatment of the third, fourth and fifth orders of 2DIMB using the constant step size strategy. In this case, the integral part of DVIDE follows the law of definite integral and transforms the integral into two parts. Hence, these parts are solved using the appropriate numerical integration based on the order of the derived method. Several numerical problems are evaluated and compared to prior methods.

Chapter 6 discusses the DVIDE of unbounded time-lag with proportional delay conditions. This chapter explores implementing the derived methods of orders three, four, and five to solve this problem. The implementation of Lagrange interpolation polynomial is needed to find the delay solution in the problems. The appropriated numerical integration adapts to the DVIDE to solve the integral part. Several numerical problems are solved and compared with the established methods.

Chapter 7 emphasizes the derivation of the two points direct diagonally implicit multistep block method for the fourth order method. This derived method directly solves the second order of DVIDE using the constant step size. Besides, this chapter presents the strategy to choose the appropriate numerical integration method for the integral part of DVIDE. All the analyses needed for the 2O2DDI have been described, including order, consistency, zero stability and convergence. The discussion of numerical results for the constant and proportional delay conditions has been investigated.

Chapter 8 summarizes the essential findings of this research study. At the same time, some potential recommendations for further research work are highlighted in this chapter.

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