GRAPHICAL PROCESSES WITH ABELIAN ROTATION SYMMETRY

By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Master of Science

May 2023

IPM 2023 1

DEDICATIONS

To my family

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

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The idea of abelian quantum rotation is applied to the well established framework of categorical quantum mechanics and we provide a novel toolbox for the simulation of finite dimensional abelian quantum rotation. Strongly complementary structures are used to give the graphical characterisation of classical aspects of abelian quantum rotation, their action on systems and the momentum observables. Weyl canonical commutation relations are identified from the axioms of strongly complementary, and the existence of dual pair of angle/momentum observables is concluded for finite dimensional abelian quantum rotation. The quantum structure of abelian quantum rotation is discussed by showing there exists a symmetry-observable duality and evolution of quantum state is described by the Eilenberg-Moore morphism. Finally, composite quantum rotational systems are constructed and it is shown that they have the synchronicity property and proved the conservation law of momentum.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PROSES-PROSES BERGRAFIK UNTUK PUTARAN ABELAN KUANTUM

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Idea putaran kuantum Abelan digunakan ke atas rangka kerja mekanik kuantum berkategori dan suatu kotak peralatan novel bagi simulasi putaran kuantum Abelan berdimensi terhingga disediakan. Struktur pelengkap kuat digunakan untuk memberi pencirian grafik untuk memahami aspek klasik putaran kuantum, tindakannya pada sistem dan pembolehcerap momentum. Hubungan penukar tertib berkanun Weyl dikenal pasti daripada aksiom pelengkap kuat, dan disimpulkan bahawa wujudnya pasangan dual pembolehcerap sudut/momentum untuk putaran kuantum Abelan berdimensi terhingga. Struktur kuantum untuk putaran kuantum Abelan dibincangkan dengan menunjukkan wujudnya dualiti antara simetri putaran dan pembolehcerap serta evolusi keadaan kuantum dapat diperihalkan oleh morfisma Eilenberg-Moore. Akhir sekali, dibangunkan sistem putaran kuantum Abelan gabungan dengan membuktikan bahawa wujudnya sifat kesegerakan dan hukum keabdian momentum.

ACKNOWLEDGEMENTS

Bismillahirahmanirahim, Alhamdulillah, praise to Allah the Almighty of God the most Gracious and the most Merciful. First of all I would like to thank God for His Blessing in the completion of this thesis. Then praises are also to Prophet Muhammad SAW who has guided us to the better life today. I would like to give a big thank you to my incredible supervisor Associate Professor Dr. Hishamuddin Zainuddin. He has always supported my research and personal well-being. He is an amazing mentor to me throughout my MSc degree. I would also thank Dr. Nurisya and Dr. Chan who have helping and motivate me to finish my MSc degree. Without their guidance I would not able to finish this thesis. Thank you to my parents Rosli Abd Rahman and Maznah Bachik for their love and prayer, motivation and supports both in moral and material, may Allah give them healthy and happiness in the world and here after. Big thanks to my sisters Nadia Qurratul Aini, Nabila Farina and Naura Qatrunnada for their hospitality during two years of pandemic. Special thank you to all the Quest group members, Ahmad Hazazi, Choong Pak Shen, Aqilah Rasat, Dr. Fauzi Umar, Dr. Umair Abdul Halim, Jessica, Amin Hamid for their helps with Latex and thesis format and sharing their experience doing research. A big thank you to Wan Nurshamimi binti Wan Jusoh for helping me in various kind of matters. In addition, the writer would like to say never give up to gain knowledge and keep on study, may Allah bless us. Finally, thanks are due to the Malaysian Government particularly Ministry of Higher Education for providing me the scholarship.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Date: 14 December 2023

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LIST OF ABBREVIATIONS

CQM	Categorical Quantum Mechanics
Set	Category of Sets
Rel	Category of Relations
Hilb	Category of Hilbert spaces
FdHilb	Category of finite Hilbert spaces
SMC	Symmetric Monoidal Category
†-SMC	Symmetric Monoidal Dagger Category
FA	Frobenius Algebra
†-FA	Dagger Frobenius Algebra
†-qSFA	Dagger quasi-special Frobenius algebra
†-SCFA	Dagger special commutative Frobenius algebra

CHAPTER 1

INTRODUCTION

1.1 General introduction

Categorical quantum mechanics (CQM) provides a powerful graphical language for quantum theory and related areas. It was done by abstracting the traditional Hilbert space formalism of much details. The argument is that abstraction is our primary tool when attempting to make sense of the physical world. It shows which ideas ought to be regarded as essential and places them in the more elegant and general formalism. In fact, an increase in the level of abstraction has been intimately tied to several paradigm shifts that have improved our understanding of nature. However, the change in point of view always find resistance from the scientific community. For instance, group theory where mathematicians always thought that it was too abstract and has no applications, but physicists found ways to counter such arguments. It is increasingly essential to contemporary physics and has a wide range of uses. Likewise, category theory can be another example of this formalism that, although it was formerly thought to be overly abstract, is now one of the key mathematics for understanding the physical world.

Moving away from ideas or presumptions that do not line up with our best theories and experiments may be one of the technique to better understand the physical world around us. While doing so is quite hard, it seems that there is one concept that we can not get rid and that concept is called composition (Coecke, 2021). This makes sense given that our brains have a finite amount of memory and that the only way we can make sense of complicated things is by putting simpler parts together. This operation is done by the process of composition. For instance is our natural language.The meaning of complex sentences is formed by combining the meanings of their constituent elements. Thus, one can take the concepts of composition as fundamental. The device that can realise this concept is category theory since the essence of categories is composition and composition yields a definition of category.

The language of symmetric monoidal categories has been used in quantum information and quantum foundations in the programme of CQM (Coecke, 2010). Within this framework one can define a category-theoretic notion of Frobenius algebra. A Frobenius algebra consists of a single object equipped with both monoid and comonoid structure such that these two structures interact in such a way that they form a Frobenius law. Frobenius algebras play an important role in CQM as they characterised orthogonal bases in *category of finite dimensional Hilbert space* **FdHilb**. Because of that, it also can be seen as non-degenerate quantum observables.

One of the important features of quantum theory is complementarity. The two bases or non-degenerate observables are complementary or unbiased if they are 'maximally incompatible'. For instance, consider we have a qubit. If we measure the qubit with spin X observable, it gives us no information at all about the Z basis. This implies the observables X and Z of a qubit are complementary. Since the basis or non-degenerate observables can be characterised as *dagger special commutative Frobenius algebra* †-SCFA, then complementary observables can be realised by interacting two *dagger Frobenius algebras* †-SCFAs to form a Hopf law.

Furthermore, the interaction of two dagger Frobenius structures could also form a bialgebra (Duncan and Dunne, 2016). Because of that the two dagger Frobenius structures form a monoid where the elements are the copyable states. With the presence of Hopf law the interacting of two dagger Frobenius structures form a group. In the category of (finite) Hilbert space, this interaction of dagger Frobenius structures are the group algebras. In the literature the group structure which comes from the interacting dagger Frobenius structures are called coherent groups.

When one studies quantum rotational system it means that the system has rotation symmetry. From mathematical point of view, the symmetry of a system is described by the action of the group on it: a set of reversible transformations of the system into itself, closed under composition and inversion. This action of coherent group can be realised in symmetric monoidal †-category (†-SMC). By using that, we can investigate the physical significance of quantum system with rotation symmetry as well as to gain some insight on what happens if we have composite systems with rotation symmetry.

1.2 Problem statement

It is hard to overstate the importance of physical system with rotation symmetry. In a very deep sense, it truly describes the non-linear motion of particle. Despite this, (abelian) quantum rotation is rarely considered directly in quantum information and foundations of quantum computing although the qubit comes from the intrinsic property of angular momentum. In this work, we aim to give a description of (abelian) quantum rotation by developing a fully diagrammatic approach based on categorical algebra.

Given such scenario, this work is dedicated to the following problems:

1. What are pair of strongly complementary observables for the system with rotation symmetry?

2. What are the classical aspects for quantum system with abelian rotation symmetry?

3. How the quantum structure behave for the system with abelian rotation symmetry?

4. How the composite quantum systems with abelian rotation symmetry behave?

1.3 Objectives of research

The essence of categorical quantum mechanics is we do not study quantum system in isolation but we group the systems together into compound systems and investigate the quantum theory from there. In this work, we try to study such quantum systems with rotation symmetry from that perspective. The objectives of this research are as the following:

1. To identify the pair of strongly complementary observables of abelian quantum rotation and prove that they satisfy Weyl canonical commutation relation.

2. To give the description of quantum system associated with abelian rotation symmetry and describes the evolution of quantum state under abelian rotation symmetry.

3. To prove the conservation law of momentum in composite quantum system with abelian rotation symmetry.

1.4 Thesis outline

This thesis consists of five chapters. We begin in the first chapter by giving the general overview of this research. Next, in chapter two we select some related literatures which motivated our interest on this work. This chapter comprises into five subtopics which includes the categorical quantum mechanics, algebraic structures in monoidal category, application of strongly complementary observables, infinite dimensional CQM and quantum symmetries in CQM. Although, our system of study is finite dimension, we also briefly explain how this method can be extended to infinite dimensional case.

In chapter three, we present the mathematical tools for process theory which will be used throughout the thesis. We first explain the definition of categories, functor and natural transformation. Next, we discuss about monoidal categories and their structures. The category that we use in this work is symmetric monoidal dagger category with duals. The important thing is that, monoidal category comes with graphical calculus and we explain about it in great details. In the next subtopic, we discuss about the algebraic structure in symmetric monoidal †-category (†-SMC) particularly the group structure in it. At the end of this chapter, we talk about the representation of group in †-SMC.

Chapter 4 explains every single details of the results on quantum system with abelian rotation symmetry. They included the construction of classical structure in abelian quantum rotation and we prove that it forms a weyl canonical commutation relation between unitary operators of angle and unitary operators of momentum. Next, we construct the quantum structure of abelian quantum rotation, the invariant state and we prove the symmetry-observable duality. Lastly, we discuss about the synchronicity of quantum systems and separability of quantum systems with abelian rotation symmetry.

The final chapter provided the conclusion of this work and also the discussion on the future research that could be taken. One of the suggestions is to look into the non-abelian group structure in symmetric monoidal dagger category to construct the full version of quantum angular momentum.

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