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RESEARCH ARTICLE

Nonlinear Fredholm Functional-Integral Equation of First Kind with Degenerate Kernel and Integral Maxima

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Abstract In this note, the problems of solving by the aid of regularization method for a nonlinear Fredholm functional-integral equation of the first kind with degenerate kernel and integral maxima are considered. In using the regularization method, we denote the nonlinear function as a new unknown function. This method we combine with the method of the degenerate kernel. Fredholm functional-integral equation of the first kind is ill-posed (non-correct) problem. We have used boundary conditions to ensure the uniqueness of the solution. We transform the implicit functional equation with integral maxima to another type of nonlinear functional-integral equation of the second kind by some integral transforms. Using the method of successive approximations and the method of compressing mapping, the theorem on single valued solvability of the problem is proved. It is obtained the necessary and sufficient conditions of existence and uniqueness of the solution of the problem with integral maxima. Two simple examples are analyzed with an exact and approximate solution.

Keywords: Fredholm functional-integral equation, nonlinear equation, degenerate kernel, integral maxima, boundary conditions, regularization, small parameter.

Introduction

Integral equations appear in many mathematical models of engineering and physical processes such as diffracting and scattering of wave problems, cracks problem in fracture mechanics. Many of the equations in these problems do not have the exact solution. Hence, numerical methods play an essential role in interpreting the behavior of their solutions. Different kind of methods are used in numerical solution of equations of mathematical physics (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). In particular, Boyd in [5] discussed the Chebyshev and Fourier spectral methods and Furano in [7] explained the polynomial approximations of solving differential equations. Shen et. al in [12] studied the nonlinear Volterra integral equations with weakly singular kernels by applying the generalized spectral Jacobi -Galerkin method. Eshkuvatov *et al*. in [13, 14] worked on the numerical approximations of integral equations with weak and strong singularity. In 2010, Behiry *et al*. developed a discrete Adomian decomposition method to solve the nonlinear Fredholm integral equations [15].

Fredholm integral equation of the first kind can be solved analytically or numerically, using various methods. Among them are the homotopy perturbation method, the Taylor series method, the collocation method, etc. The Adomian decomposition method (see [16, 17]) has been widely

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used in the theory of linear and nonlinear problems for differential and integral equations efficiently. The solution to this problem is of the form of an infinite series in which each term can be easily determined. In [18], Cherruault et. al discussed the rapid convergence of the series obtained by this method. Moreover, in [19, 20, 21, 22] proposed a modified technique of Adomian decomposition method that accelerates the rapid convergence of decomposition series solution. In [23], Hadamard gave a definition of the well-posed problem. In [24] Phillips established the method of regularization in dealing with the linear Fredholm integral equations of the first kind. In [25] the noncorrect problems of mathematical physics and analysis are studied. By introducing a new unknown function and a small positive parameter, the regularization method reduces ill-posed problem to the well-posed problem.

In this paper, the problems of single valued solvability for a first kind nonlinear Fredholm functional-integral equation with the degenerate kernel and integral maxima are studied. The functional-integral equation is transformed to another type of equation, which is convenient to solve. So, utilizing the regularization method together with the degenerate kernel method and using some integral transforms, an implicit equation with integral maxima is obtainable. To ensure the uniqueness of solution the boundary conditions are utilized. Then, using the successive approximations method, the theorem on single valued solvability is proved. We note that in [26, 27] are considered equations with some particular cases of construction of maxima. In contrast to these works, in our article we study equations with nonlinear integral maxima.

Formulation of the problem

On the given segment $[0;T]$, the following form of nonlinear Fredholm integral equation with integral maxima

$$f(t)=\int_0^T K(t,s)F\left(s,u(s),\max\left\{u(\tau):\tau\in\left[h_1\left(s,\int_0^T\Theta_1(s,\theta)u(\theta)d\theta\right);h_2\left(s,\int_0^T\Theta_2(s,\theta)u(\theta)d\theta\right)\right]\right\}\right)ds \qquad (1)$$

is considered. Let be given the following boundary value conditions with known initial and final valued functions

$$\left[\begin{array}{l} u(0)=0, \\ u(t)=\varphi_1(t),\ t\in(-\infty;0], \\ u(T)=\varphi_0=const, \\ u(t)=\varphi_2(t),\ t\in[T;\infty), \end{array}\right. \qquad (2)$$

where $0<T$ is given real number, $-\infty<h_i<\infty$, $\int_0^T|\Theta_i(s,\theta)|d\theta<\infty$, $i=1,2$, $\varphi_1(t)\in C(-\infty;0]\cap Bnd(T_0)$, $\varphi_2(t)\in C[T;\infty)\cap Bnd(T_0)$, $K(t,s)=\sum_{i=1}^k a_i(t)b_i(s)$, $0\neq a_i(t), b_i(s)\in C[0;T]$. Here it is assumed that each of the systems of functions $a_i(t)$, $i=\overline{1,k}$, and $b_i(s)$, $i=\overline{1,k}$, are linearly independent. Here by the symbol $Bnd(T_0)$ we denote the boundless of functions by the given positive number T_0. Moreover, the conditions of continuous gluing at the end points of the segment are also valid: $\varphi_1(0)=u(0)$, $\varphi_2(T)=u(T)$

Regularization and Degenerate Kernel Methods

We proceed to transform the integral equation (1). Taking into account the degeneracy of the kernel, equation (1) is written in the following form

$$f(t)=\int_0^T\sum_{i=1}^k a_i(t)\,b_i(s)F\left(s,u(s),\max\left\{u(\tau):\tau\in\left[h_1\left(s,\int_0^T\Theta_1(s,\theta)u(\theta)d\theta\right);h_2\left(s,\int_0^T\Theta_2(s,\theta)u(\theta)d\theta\right)\right]\right\}\right)ds. \qquad (3)$$

Denoting the non-linear integrand function as a new unknown function

$$\vartheta(t)=F\left(t,u(t),\max\left\{u(\tau):\tau\in\left[h_1\left(t,\int_0^T\Theta_1(t,s)u(s)ds\right);h_2\left(t,\int_0^T\Theta_2(t,s)u(s)ds\right)\right]\right\}\right) \qquad (4)$$

and noting dependence of a new unknown function from the small positive parameter ε and

$$\lim_{\varepsilon\to 0}\vartheta(t,\varepsilon)=\vartheta(t), \tag{5}$$

we obtain from integral equation (3) approximation linear Fredholm second kind integral equation with small parameter

$$\varepsilon\vartheta_{\varepsilon}(t,\varepsilon)=f(t)-\int_0^T\sum_{i=1}^k a_i(t)\,b_i(s)\vartheta_{\varepsilon}(s,\varepsilon)\,ds. \tag{6}$$

By the aid of a new unknown quantity

$$\alpha_i=\int_0^T b_i(s)\vartheta(s,\varepsilon)\,ds, \tag{7}$$

the integral equation (6) can be rewritten as

$$\vartheta(t,\varepsilon)=\frac{1}{\varepsilon}\left[f(t)-\sum_{i=1}^k a_i(t)\alpha_i\right]. \tag{8}$$

Substituting the presentation (8) into (7), we obtain the system of linear equations (SLE)

$$\alpha_i+\sum_{j=1}^k\alpha_j A_{ij}=B_i,\ \ i=\overline{1,k}, \tag{9}$$

where

$$A_{ij}=\frac{1}{\varepsilon}\int_0^T b_i(s)a_j(s)\,ds,\ \ B_i=\frac{1}{\varepsilon}\int_0^T b_i(s)f(s)\,ds. \tag{10}$$

To solve the SLE (9), we consider the following matrices:

$$P=\begin{pmatrix}1+A_{11} & A_{12} & \cdots & A_{1k}\\ A_{21} & 1+A_{22} & \cdots & A_{2k}\\ \cdots & \cdots & \cdots & \cdots\\ A_{k1}^m & A_{k2} & \cdots & 1+A_{kk}\end{pmatrix},$$

$$P_i=\begin{pmatrix}1+A_{11} & \ldots & A_{1(i-1)} & B_1 & A_{1(i+1)} & \ldots & A_{1k}\\ A_{21} & \ldots & A_{2(i-1)} & B_2 & A_{2(i+1)} & \ldots & A_{1k}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ A_{k1} & \ldots & A_{k(i-1)} & B_k & A_{k(i+1)} & \ldots & 1+A_{kk}\end{pmatrix},\ \ i=\overline{1,k}.$$

The solutions of SLE (9) are written as

$$\alpha_i=\frac{\Delta_i}{\Delta},\ \ i=\overline{1,k}, \tag{11}$$

if the non-degeneracy condition of the Fredholm determinant

$$\Delta=\det P\neq 0 \tag{12}$$

is fulfilled, where $\Delta_i=\det P_i,\ i=\overline{1,k}$. Substituting the solutions (12) of the SLE (9) into presentation (8), we derive

$$\vartheta(t,\varepsilon)=\frac{1}{\varepsilon}\left[f(t)-\sum_{i=1}^k a_i(t)\frac{\Delta_i}{\Delta}\right]. \tag{13}$$

We cannot claim that all the functions presented in formula (13) are what we are looking for. So, we impose conditions under which presentation (13) will become interesting for us. By virtue of the quantities (10), we will suppose that the following conditions are fulfilled:

$$f(t)=\sum_{i=1}^k a_i(t)c_i,\ \ c_i-\frac{\Delta_i}{\Delta}=\varepsilon C_i, \tag{14}$$

where c_i and C_i $(i=\overline{1,k})$ are some constants.

Then, taking into account limit passing condition (5) as $\varepsilon \to 0$ in the presentation (13), we obtain

$$\vartheta(t) = \sum_{i=1}^{k} C_i a_i(t). \tag{15}$$

Now the function $\vartheta(t)$ is defined by the equation (15). Thus, we rewrite the implicit functional integral equation with integral maxima (1) in the form of (4):

$$0 = G\left(t, u(t), \max\left\{u(\tau) : \tau \in \left[h_1\left(t, \int_0^T \Theta_1(t,s)u(s)ds\right); h_2\left(t, \int_0^T \Theta_2(t,s)u(s)ds\right)\right]\right\}\right) \tag{16}$$

with

$$\begin{cases} u(0) = 0, \\ u(t) = \varphi_1(t),\ t \in (-\infty; 0], \\ u(T) = \varphi_0 = const, \\ u(t) = \varphi_2(t),\ t \in [T; \infty), \end{cases}$$

where $G = F - \vartheta$.

Equation (16) is a non-linear functional equation with integral maxima. To solve it, we propose a new and efficient technique that allows us to determine the unique solution of the equation (1) with boundary data (2) under the above indicated conditions (11) and (14). In some simplest cases, the nonlinear equation (16) is analytically solved by an elementary transformation and is also easily solved using numerical methods. We will show these with examples. So, the following two examples are presented to show how these methods are practically working in solving the Fredholm integral equations.

Example 1. Applying regularization method in combination with the method of degenerate kernel, we consider the following simple integral equation of first kind

$$9t = \int_0^1 \frac{4ts}{e^2+1}\left(e^s + 2u^2(s)\right)^2 ds. \tag{17}$$

Firstly, we use the regularization method by denoting $\vartheta(t) = \left(e^t + 2u(t)\right)^2$. Then from the nonlinear integral equation (17) we come to the second kind linear integral equation

$$\vartheta(t,\varepsilon) = \frac{1}{\varepsilon}\left[9t - \int_0^1 \frac{4ts}{e^2+1}\vartheta(s,\varepsilon)\,ds\right]. \tag{18}$$

Now we use degenerate method by denoting a new unknown quantity

$$\alpha = \int_0^1 s\,\vartheta(s,\varepsilon)\,ds, \tag{19}$$

and, by virtue of denotation (19), from the integral equation (18), we obtain the following presentation

$$\vartheta(t,\varepsilon) = \frac{t}{\varepsilon}\left[9 - \frac{4}{e^2+1}\alpha\right]. \tag{20}$$

We substitute the presentation (20) into designation (19), which produces the following linear algebraic equation

$$\alpha = \frac{1}{3\varepsilon}\left[9 - \frac{4\alpha}{e^2+1}\right].$$

This linear algebraic equation is unique solvable. So, from this, it follows that

$$\alpha = \frac{9(e^2+1)}{3\varepsilon(e^2+1)+4}. \tag{21}$$

Substituting the value (21) of α into presentation (20), yields

$$\vartheta(t,\varepsilon)=\frac{t}{\varepsilon}\left[9-\frac{4}{e^2+1}\cdot\frac{9(e^2+1)}{3\varepsilon(e^2+1)+4}\right]=\frac{27(e^2+1)t}{3\varepsilon(e^2+1)+4}. \tag{22}$$

Applying the limit $\varepsilon\to 0$ in presentation (22), we obtain the following desire function

$$\vartheta(t)=\frac{27(e^2+1)t}{4}. \tag{23}$$

By using inverse transformation, we obtain the following implicit functional equation, given by

$$\left(e^t+2u^2(t)\right)^2=\frac{27(e^2+1)t}{4}. \tag{24}$$

It is not difficult to observe that this equation has a solution given by

$$u(t)=\pm\sqrt{\frac{3}{4}\sqrt{3\left(e^2+1\right)t}-\frac{1}{2}e^t}. \tag{25}$$

It is obvious that the function (25) satisfies the given integral equation (17). However, it is noted that this equation has also other solution, i.e. $u(t)=\pm e^{\frac{t}{2}}$. Indeed, it is not difficult to proof that this function is also solution of the equation (17). Indeed,

$$9t=\int_0^1\frac{4ts}{e^2+1}\left(e^s+2u^2(s)\right)^2ds=\int_0^1\frac{4ts}{e^2+1}\left(e^s+2e^s\right)^2ds=\frac{36t}{e^2+1}\int_0^1 s\,e^{2s}ds=9t. \tag{26}$$

Thus, we reached $9t=9t$.
We now solve this example equation, using the Adomian Decomposition Method (ADM). First, we rewrite equation (18) in the following form

$$\varepsilon\vartheta(t,\varepsilon)=9t-\frac{4}{e^2+1}\cdot\frac{t}{\varepsilon}\int_0^1 s\varepsilon\vartheta(s,\varepsilon)\,ds. \tag{27}$$

The solution of integral equation (27) can be written in the series form

$$\vartheta(t,\varepsilon)=\vartheta_0(t,\varepsilon)+\sum_{m=1}^{\infty}\vartheta_m(t,\varepsilon). \tag{28}$$

By substituting the presentation (28) into the equation (27), yields

$$\varepsilon\vartheta_0(t,\varepsilon)+\varepsilon\vartheta_1(t,\varepsilon)+\varepsilon\vartheta_2(t,\varepsilon)+\cdots=9t-\frac{4}{e^2+1}\cdot\frac{t}{\varepsilon}\int_0^1 s\left[\varepsilon\vartheta_0(t,\varepsilon)+\varepsilon\vartheta_1(t,\varepsilon)+\varepsilon\vartheta_2(t,\varepsilon)+\cdots\right]ds. \tag{29}$$

Hence, we produce the following recurrence relation as follows

$$\begin{cases}\varepsilon\vartheta_0(t,\varepsilon)=9t,\\ \varepsilon\vartheta_m(t,\varepsilon)=-\dfrac{4}{e^2+1}\cdot\dfrac{t}{\varepsilon}\displaystyle\int_0^1 s\left(\varepsilon\vartheta_{m-1}(s,\varepsilon)\right)ds,\ m=1,2,3,...\end{cases} \tag{30}$$

From the iterative process (30), it follows that

$$\begin{cases}\varepsilon\vartheta_0(t,\varepsilon)=9t,\\ \varepsilon\vartheta_1(t,\varepsilon)=-\dfrac{4}{e^2+1}\cdot\dfrac{t}{\varepsilon}\displaystyle\int_0^1 s\left(9s\right)ds=-\dfrac{4}{3\varepsilon\left(e^2+1\right)}\cdot 9t,\\ \varepsilon\vartheta_2(t,\varepsilon)=-\dfrac{4}{e^2+1}\cdot\dfrac{t}{\varepsilon}\displaystyle\int_0^1 s\left(-\dfrac{4}{3\varepsilon\left(e^2+2\right)}\cdot 9s\right)ds=\left[\dfrac{4}{3\varepsilon\left(e^2+1\right)}\right]^2 9t.\end{cases} \tag{31}$$

By continuing this procedure, we obtain

$$\varepsilon\vartheta_m(t,\varepsilon)=(-1)^m\left[\frac{4}{3\varepsilon\left(e^2+1\right)}\right]^m 9t. \tag{32}$$

Next, introducing the notation

$$q=\frac{4}{3\varepsilon\left(e^2+1\right)} \tag{33}$$

and substituting presentation of solution (32) into the series (28), yields

$$\varepsilon\vartheta(t,\varepsilon)=9t(1-q)\left(1+q^2+q^4+q^6+\cdots\right)=9t\frac{1-q}{1-q^2}=\frac{9t}{1+q}=\frac{27\varepsilon\left(e^2+1\right)t}{4+3\varepsilon\left(e^2+1\right)}.$$

Hence, we obtain

$$\vartheta(t,\varepsilon)=\frac{27\left(e^2+1\right)t}{4+3\varepsilon\left(e^2+1\right)},$$

which yields

$$\lim_{\varepsilon\to 0}\vartheta(t,\varepsilon)=\lim_{\varepsilon\to 0}\frac{27\left(e^2+1\right)t}{4+3\varepsilon\left(e^2+1\right)}=\frac{27\left(e^2+1\right)t}{4}=\vartheta(t).$$

This solution is identical with the exact solution of linear equation (18). To determine the exact solution of the problem (17), we perform similar process as in equations (23)-(25).

Example 2. Consider another integral equation

$$\frac{2t^2+1}{5}=\int_0^1\left(2t^2+1\right)s\,u^3(s)\,ds, \tag{34}$$

with given initial value condition $u(0)=1$.

We determine a unique solution of the integral equation (34), satisfying the given initial value condition $u(0)=1$. Let $\vartheta(t)=u^3(t)$. Then, we obtain the linear integral equation of the second kind with a small parameter, given by

$$\vartheta(t,\varepsilon)=\frac{1}{\varepsilon}\left[\frac{2t^2+1}{5}-\int_0^1\left(2t^2+1\right)s\,\vartheta(s,\varepsilon)\,ds\right]. \tag{35}$$

As in first example, we make the following designation

$$\alpha=\int_0^1 s\,\vartheta(s,\varepsilon)\,ds. \tag{36}$$

Then, from equation (35), we reached the following presentation

$$\vartheta(t,\varepsilon)=\frac{1}{\varepsilon}\left(2t^2+1\right)\left(\frac{1}{5}-\alpha\right). \tag{37}$$

We now substitute the presentation (37) into designation (36). After calculating the obtained integral, we come to solve the following algebraic equation

$$\alpha=\frac{1}{\varepsilon}\left(\frac{1}{5}-\alpha\right).$$

The solution of this linear equation is

$$\alpha=\frac{1}{5\varepsilon+5}. \tag{38}$$

So, substituting solution (38) into the representation (37), yields

$$\vartheta(t,\varepsilon)=\frac{5\left(2t^2+1\right)}{5\varepsilon+5}. \tag{39}$$

Applying the limit $\varepsilon \to 0$ to the function (39), we have

$$\vartheta(t)=2t^2+1. \tag{40}$$

Moreover, upon applying the inverse transform, the following implicit functional equation is obtainable;

$$u^3(t)=2t^2+1, \tag{41}$$

which yields solutions

$$u(t)=\sqrt[3]{2t^2+1}. \tag{42}$$

This solution satisfies the given integral equation (34). However, it also has another solution: $u(t)=t$. We now need to determine the unique solution of the equation (34). Thus, we apply the given initial value condition $u(0)=1$ and obtain that the unique solution of this integral equation is $u(t)=\sqrt[3]{2t^2+1}$. This solution satisfies the given initial value condition $u(0)=1$.

We will now solve Example 2, using ADM. First, we rewrite equation (35) in the form

$$\varepsilon\vartheta(t,\varepsilon)=\frac{2t^2+1}{5}-\frac{1}{\varepsilon}\int_0^1\left(2t^2+1\right)s\left(\varepsilon\vartheta(s,\varepsilon)\right)ds. \tag{43}$$

Searching solutions of the series form is similar to equation (28) and substituting it into linear equation (43) yields

$$\varepsilon\vartheta_0(t,\varepsilon)+\varepsilon\vartheta_1(t,\varepsilon)+\varepsilon\vartheta_2(t,\varepsilon)+\cdots=\frac{2t^2+1}{5}-\frac{2t^2+1}{\varepsilon}\int_0^1 s\left[\varepsilon\vartheta_0(t,\varepsilon)+\varepsilon\vartheta_1(t,\varepsilon)+\varepsilon\vartheta_2(t,\varepsilon)+\cdots\right]ds,$$

which gives

$$\begin{cases}\varepsilon\vartheta_0(t,\varepsilon)=\dfrac{2t^2+1}{5},\\ \varepsilon\vartheta_m(t,\varepsilon)=-\dfrac{2t^2+1}{\varepsilon}\displaystyle\int_0^1 s\left(\varepsilon\vartheta_{m-1}(s,\varepsilon)\right)ds.\end{cases} \tag{44}$$

From the successive approximations (44), we obtain

$$\begin{cases}\varepsilon\vartheta_0(t,\varepsilon)=\dfrac{2t^2+1}{5},\\ \varepsilon\vartheta_1(t,\varepsilon)=-\dfrac{2t^2+1}{5\varepsilon}\displaystyle\int_0^1 s\frac{2s^2+1}{5}ds=-\frac{2t^2+1}{5}\cdot\frac{1}{5\varepsilon},\\ \varepsilon\vartheta_2(t,\varepsilon)=-\dfrac{2t^2+1}{5\varepsilon}\displaystyle\int_0^1\frac{2s^2+1}{5}\cdot\frac{s}{5\varepsilon}ds=\frac{2t^2+1}{5}\cdot\left(\frac{1}{5\varepsilon}\right)^2,\\ \vdots\\ \varepsilon\vartheta_m(t,\varepsilon)=(-1)^m\dfrac{2t^2+1}{5}\cdot\left(\dfrac{1}{5\varepsilon}\right)^m.\end{cases} \tag{45}$$

Introducing the notation

$$q=\frac{1}{5\varepsilon} \tag{46}$$

and using the series (28) and iterations (45), we have

$$\varepsilon\vartheta(t,\varepsilon)=\frac{2t^2+1}{5}\left(1-q+q^2-q^3+\cdots\right)=\frac{2t^2+1}{5}(1-q)\left(1+q^2+q^4+\cdots\right)=$$

$$=\frac{2t^2+1}{5}\frac{1-q}{1-q^2}=\frac{2t^2+1}{5}\frac{1}{1+q}=\frac{2t^2+1}{5}\frac{1}{1+\frac{1}{5\varepsilon}}=\frac{2t^2+1}{1+5\varepsilon}\varepsilon. \quad (47)$$

Applying the limit $\varepsilon \to 0$ in (47), we obtain the solution for integral equation (35):

$$\vartheta(t,\varepsilon)=\frac{2t^2+1}{1+5\varepsilon},$$

which yields

$$\lim_{\varepsilon\to 0}\vartheta(t,\varepsilon)=\lim_{\varepsilon\to 0}\frac{2t^2+1}{1+5\varepsilon}=2t^2+1=\vartheta(t).$$

This solution is identical with the exact solution of linear equation (18). To determine the exact solution of the problem (17), we perform similar process as in equations (23)-(25). Thus, ADM gives exact solution of the linear equation (35). To obtain the exact solution of the problem (34), we follow the same steps as we did in equations (40)-(42).

Transform into Nonlinear Functional Integral Equation

In our complicated case, it is impossible to solve the equation (16) by the elementary ways. So, we perform the following strategy: on the segment $[0;T]$ we replace the problem (1), (2) with the non-linear functional-integral equation with integral maxima (see [28])

$$u(t)=\Im(t;u)\equiv\int_0^t H(t,s)u(s)ds$$
$$+\left[u(t)+G\left(t,u(t),\max\left\{u(\tau):\tau\in\left[h_1\left(t,\int_0^T\Theta_1(t,s)u(s)ds\right);h_2\left(t,\int_0^T\Theta_2(t,s)u(s)ds\right)\right]\right\}\right)\right]\cdot\exp\{-\psi(t)\}$$
$$+\int_0^t K_0(s)\exp\{-\psi(t,s)\}\left\{u(t)-u(s)+G\left(t,u(t),\max\left\{u(\tau):\tau\in\left[h_1\left(t,\int_0^T\Theta_1(t,s)u(s)ds\right);h_2\left(t,\int_0^T\Theta_2(t,s)u(s)ds\right)\right]\right\}\right)\right.$$
$$\left.-G\left(s,u(s),\max\left\{u(\tau):\tau\in\left[h_1\left(s,\int_0^T\Theta_1(s,\theta)u(\theta)d\theta\right);h_2\left(s,\int_0^T\Theta_2(s,\theta)u(\theta)d\theta\right)\right]\right\}\right)\right\}ds,\ t\in[0;T], \quad (48)$$

where

$$H(t,s)=K_0(s)\exp\{-\psi(t,s)\}-\int_s^t K_0(\theta)\exp\{-\psi(t,\theta)\}K_0(\theta,s)d\theta, \quad (49)$$

$$\psi(t,s)=\int_s^t K_0(\theta)d\theta,\ \psi(t,0)=\psi(t),\ t\in[0;T].$$

For the function $\psi(t,s)$ it is true that $\psi(t,s)=\psi(t)-\psi(s),\ \psi(0)=0$. By the solution of the equation (1) we understand a continuous function $u(t)$ on the segment $[0;T]$, satisfying equation (1) with the given boundary value conditions (2) and the following Lipschitz condition:

$$\|u(t)-u(s)\|\le L_0\,|t-s|, \quad (50)$$

where $0<L_0=const,\ \|u(t)\|=\max_{0\le t\le T}|u(t)|$.

Theorem. Let the conditions (12), (14) and (50) be satisfied and

1) $\left\|G\left(t,u(t),v(t)\right)\right\|\le M_0;$

2) $\left| G\big(t,u_1(t),v_1(t)\}\big) - G\big(t,u_2(t),v_2(t)\}\big) \right| \le L_1(t)\big(\left| u_1(t)-u_2(t) \right| + \left| v_1(t)-v_2(t) \right|\big);$

3) $\left| h_i\big(t,v_1(t)\}\big) - h_i\big(t,v_2(t)\}\big) \right| \le L_{1+i}(t)\left| v_1(t)-v_2(t) \right|,\ i=1,2;$

4) $0 < M_0 = const,\ \ 0 < L_1(t) \in C[0;T];$

5) $\rho < 1,$ where

$$\rho = \max_{0\le t\le T}\left[\| K_0(t,s) \| \int_0^t Q(t,s)\,ds + \left\| 1 + L_1(t)\left[1 + L_2(t)\int_0^T \left| \Theta_1(t,s) \right| ds + L_3(t)\int_0^T \left| \Theta_2(t,s) \right| ds \right] \right\| Q(t,0) \right],$$

$$Q(t,s) = \exp\left\{-\psi(t,s)\right\} + 2\int_s^t K_0(\theta)\exp\left\{-\psi(t,\theta)\right\} d\theta.$$

Then, the functional-integral equation (48) with boundary value conditions (2) has a unique solution on the segment $[0;T]$. Moreover, the solution can be founded by the following Picard iteration process

$$\begin{cases} \begin{bmatrix} u_0(t) = \varphi_1(t),\ t\in(-\infty;0], \\ u_0(t) = \varphi_0,\ t\in[0;T], \\ u_0(t) = \varphi_2(t),\ t\in[T;\infty), \end{bmatrix} \\ \begin{bmatrix} u_{n+1}(t) = \varphi_1(t),\ t\in(-\infty;0], \\ u_{n+1}(t) = \Im(t;u_n),\ n\in N,\ \ t\in[0;T], \\ u_{n+1}(t) = \varphi_2(t),\ t\in[T;\infty). \end{bmatrix} \end{cases}$$

Proof. For the function $H(t,s)$, given by formula (49), is true the following estimate

$$\left| H(t,s) \right| \le \| K_0(t,s) \| \exp\left\{-\psi(t,s)\right\} + \int_s^t \| K_0(t,\theta) \| K_0(\theta)\exp\left\{-\psi(t,\theta)\right\} d\theta \le \| K_0(t,s) \| \cdot Q(t,s), \tag{51}$$

So, for the first approximation is true the following estimate

$$\| u_0(t) \| \le \max\left\{ \left| \varphi_0 \right|; \max_{-\infty<t\le 0} \left| \varphi_1(t) \right|; \max_{T\le t<\infty} \left| \varphi_2(t) \right| \right\} = \Delta_0 < \infty. \tag{52}$$

By virtue of first condition of theorem and estimates (51) and (52), for the first approximation $u_1(t)$ of Picard process we obtain the estimate

$$\begin{aligned} \left| u_1(t) \right| \le & \int_0^t \| H(t,s) \| \cdot \| u_0(s) \| ds \ \ + \int_0^t K_0(s)\exp\left\{-\psi(t,s)\right\}\Big[\| u_0(t) - u_0(s) \| \\ & + \left[\| u_0(t) \| + \left\| G\left(t, u_0(t), \max\left\{ u_0(\tau) : \tau \in \left[h_1\left(t, \int_0^T \Theta_1(t,s)u_0(s)ds \right); h_2\left(t, \int_0^T \Theta_2(t,s)u_0(s)ds \right) \right] \right\} \right) \right\| \right] \exp\{-\psi(t)\} \\ & +2\left\| G\left(t, u_0(t), \max\left\{ u_0(\tau) : \tau \in \left[h_1\left(t, \int_0^T \Theta_1(t,s)u_0(s)ds \right); h_2\left(t, \int_0^T \Theta_2(t,s)u_0(s)ds \right) \right] \right\} \right) \right\| \Big] ds \\ & \le \Delta_0 \| K_0(t,s) \| \int_0^t Q(t,s)\,ds + \Delta_1 Q(t,0), \end{aligned} \tag{53}$$

where

$$\Delta_1 = \max\left\{ \Delta_0 + M_0;\ \| K_0(t,s) \| \cdot \left(L_0 T + 2M_0 \right) \right\}.$$

Analogously to estimate (53), we have

$$|u_{n+1}(t)-u_n(t)|\leq\int_0^t\|K_0(t,s)\|\cdot Q(t,s)\cdot\|u_n(s)-u_{n-1}(s)\|ds$$
$$+\left(1+L_1(t)\left[1+L_2(t)\int_0^T|\Theta_1(t,s)|ds+L_3(t)\int_0^T|\Theta_2(t,s)|ds\right]\right)\cdot\|u_n(t)-u_{n-1}(t)\|\cdot\exp\{-\psi(t)\}$$
$$+\int_0^t\|K_0(t,s)\|\cdot\exp\{-\psi(t,s)\}\left\{1+L_1(s)\left[1+L_2(s)\int_0^T|\Theta_1(s,\theta)|d\theta+L_3(s)\int_0^T|\Theta_2(s,\theta)|d\theta\right]\right\}\cdot\|u_n(s)-u_{n-1}(s)\|ds$$
$$\leq\|u_n(t)-u_{n-1}(t)\|\left[\|K_0(t,s)\|\int_0^t Q(t,s)\,ds+\left\|1+L_1(t)\left[1+L_2(t)\int_0^T|\Theta_1(t,s)|ds+L_3(t)\int_0^T|\Theta_2(t,s)|ds\right]\right\|Q(t,0)\right].\quad(54)$$

In choosing the arbitrary function $K_0(t)$ we take into account that

$$\psi(t,s)=\int_s^t K_0(\theta)\,d\theta\gg 1,\ t\in[0;T].$$

So, it is easy to check, that $\exp\{-\psi(t)\}\ll 1$. Thus, the functions $H(t,s)$ and $Q(t,s)$ are small. Then the functions $L_1(t),\ L_2(t),\ \Theta_1(t,s),\ \Theta_2(t,s)$ can be chosen such that

$$\rho=\max_{0\leq t\leq T}\left[\|K_0(t,s)\|\int_0^t Q(t,s)\,ds+\left\|1+L_1(t)\left[1+L_2(t)\int_0^T|\Theta_1(t,s)|ds+L_3(t)\int_0^T|\Theta_2(t,s)|ds\right]\right\|Q(t,0)\right]<1.\quad(55)$$

Consequently, from the estimate (54) we have

$$\|u_{n+1}(t)-u_n(t)\|\leq\rho\cdot\|u_n(t)-u_{n-1}(t)\|.\quad(56)$$

According to the (55), we have that $\rho<1$. So, it follows from the estimate (56) that the integral operator on the right-hand side of (48) is a compressed mapping. Hence, by estimates (52), (53) and (56) we deduce that the integral equation (48) with integral maxima and boundary value conditions (2) has a unique solution on the segment $C[0;T]$. The theorem is proved.

Conclusion

The problems of solvability and construction of solutions of a non-linear Fredholm functional-integral equation (1) of the first kind with degenerate kernel and integral maxima are studied. To guarantee the uniqueness of the solution we apply the boundary value conditions (2). The regularization method incorporated with the degenerate kernel method is employed. The problem is reduced to the implicit functional equation of second kind with integral maxima (48). The one value solvability of the problem is proved.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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