MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 4, pp. 1215–1221 (2023)

Penalty method for pricing American-style Asian option with jumps diffusion process

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(Received 26 September 2023; Revised 8 November 2023; Accepted 9 November 2023)

American-style options are important derivative contracts in today's worldwide financial markets. They trade large volumes on various underlying assets, including stocks, indices, foreign exchange rates, and futures. In this work, a penalty approach is derived and examined for use in pricing the American style of Asian option under the Merton model. The Black–Scholes equation incorporates a small non-linear penalty factor. In this approach, the free and moving boundary imposed by the contract's early exercise feature is removed in order to create a stable solution domain. By including Jump-diffusion in the models, they are able to capture the skewness and kurtosis features of return distributions often observed in several assets in the market. The performance of the schemes is investigated through a series of numerical experiments.

Keywords: *American option; Asian option; jumps-diffusion process; penalty method.*
2010 MSC: 60J65, 60J75, 91G20 **DOI:** 10.23939/mmc2023.04.1215

1. Introduction

The American-style Asian option is a kind of financial derivative that grants the holder the right, but not the responsibility, to buy or sell an underlying asset at a certain price, known as the strike price, on or before a particular date, known as the expiry date. Meanwhile, the Asian option differs from a standard option in that the payoff is based on the average price of the underlying asset over a specific period, rather than just the price of the asset at the expiration date.

Instead of just being able to exercise the option at the expiration date as in a European-style of Asian option, the holder of an American-style Asian option has the right to exercise the option at any time before the expiration date. This added flexibility can make American-style Asian options more valuable than their European-style counterparts, but it also makes them more complex to price and hedge. American-style Asian options are commonly used in financial markets as a tool for managing risk and speculating on price movements. They are particularly useful in markets where prices are volatile and difficult to predict, such as the stock market or commodity markets.

Basically, the equivalence between the American and European call prices is evident within the Black–Scholes framework. However, a disparity arises in the price of put options, which persists within Merton's conceptualization of a numerical resolution for the challenge of pricing American options has commanded considerable scholarly attention during the previous decade. Since analytical solutions to Black–Scholes models of American option problems are challenging to find. The partial integro-differential equation (PIDE) lends itself to various computational methods, such as the employment of multinomial trees as described by [1] and the utilization of the utilization successive over-relaxation method (PSOR) as elaborated by [2]. Furthermore, [10] has documented a pricing framework for Asian options with early exercise features under a jump-diffusion process, employing the Monte Carlo simulation technique.

Recently, the applications of the Penalty method to determine the price of American-style options have become increasingly attractive and popular. Moreover, [3] has carefully studied the advantages and disadvantages of three models: the stochastic volatility approach, the jump-diffusion model approach, and the deterministic volatility function approach. The jump-diffusion model has emerged as exceptionally versatile. [6] demonstrated that the accuracy of the Black–Scholes model should be improved by including discontinuous jump processes into the conventional stochastic process of geometric Brownian motion (GBM). Notably, [7] pioneered such models for the study of option valuation, and some studies that included jumps in the pricing model are [12–14]. Hence, we focused on developing a penalty method for solving a PIDE problem. This mathematical model is the cornerstone for valuing American-style Asian options featuring a jump-diffusion process. The methods draw upon the linear complementarity problem and variational inequalities to yield the solution.

Essentially, the concept underlying the penalty method for evaluate the American-style of Asian option is comparable to the strategy presented in [4] and [5] for American options with the jump-diffusion process. The pricing of American-style Asian options with a jumps diffusion process using the penalty method involves two steps. Firstly, the continuous component of the underlying asset price process is modeled using a diffusion process, such as a geometric Brownian motion or a jump-diffusion process. Secondly, the discontinuous component of the underlying asset price process is modeled using a jump process, such as a compound Poisson process.

Since almost all options traded on exchanges include the American-style option with early exercises feature, knowing that pricing of these options using a jump-diffusion model is practical and useful. Computing the price of the American-style option is indirectly involves calculating the optimal exercise policy where a decision maker determines when to terminate the system to optimize a specific objective. Since the literature evaluating American-style Asian option pricing models with jumps in assets and volatility is somewhat limited. Therefore, this study aims to contribute to this stream of literature.

2. Mathematical model of Asian option

To derive a pricing formula for Asian options, we consider S and A independent variables. This is true because the value of S is independent of its past. Assuming that the option's price is a function of S, A, and t, let us denote it $V(S,A,t)$. Considering that A must satisfy the stochastic differential equation. The payoff function of an Asian option is therefore provided as follows:

$$A_T = \frac{1}{T}\int_0^T S_t\,dt, \tag{1}$$

where $\max(A_T - S, 0)$.

Assuming dt is a small time step, it holds to the first order as follows:

$$\begin{aligned} A(t+dt) &= A_t + dA_t \\ &= \frac{1}{t+dt}\int_0^{t+dt} S_\tau\,d\tau \\ &= \frac{1}{t+dt}\int_0^{t} S_\tau\,d\tau + \frac{S_t}{t+dt}\,dt \\ &= A_t - \frac{A_t}{t}\,dt + \frac{S_t}{t}\,dt. \end{aligned}$$

Thus,

$$dA = \frac{S-A}{t}\,dt. \tag{2}$$

The following partial differential equation (PDE) is fulfilled by applying the two-dimensional Ito's Lemma to the variable $V(S,A,t)$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{S-A}{t}\frac{\partial V}{\partial A} - rV = 0. \tag{3}$$

Due to two spatial dimensions of Equation (3), its solution will require more calculations than the Black–Scholes equation. Since it is possible to reduce the dimensionality for the floating-strike Asian options by a change of variables. Hence, consider change of variables defined as follows:

$$H=\frac{V}{S}\quad\text{and}\quad R=\frac{A}{S}.$$

We obtain the following partial derivatives of H with respect to S and A in terms of $\frac{\partial H}{\partial R}$ using the chain rule:

$$\begin{aligned}
\frac{\partial H}{\partial S}&=\frac{\partial H}{\partial R}\frac{\partial R}{\partial S}\frac{A}{S}=\frac{A}{S^2}\frac{\partial H}{\partial R}=-\frac{R}{S}\frac{\partial H}{\partial R},\\
\frac{\partial^2 H}{\partial S}&=\frac{\partial}{\partial S}\Big(-\frac{A}{S^2}\frac{\partial H}{\partial R}\Big)=\frac{2A}{S^3}\frac{\partial H}{\partial R}-\frac{A}{S^2}\frac{\partial R}{\partial S}\frac{\partial^2 H}{\partial R^2}=\frac{2R}{S^2}\frac{\partial H}{\partial R}+\frac{R^2}{S^2}\frac{\partial^2 H}{\partial R^2},\\
\frac{\partial H}{\partial A}&=\frac{\partial H}{\partial R}\frac{\partial R}{\partial A}=\frac{\partial}{\partial A}\left(\frac{A}{S}\right)\frac{\partial H}{\partial R}=\frac{1}{S}\frac{\partial H}{\partial R}.
\end{aligned}$$

Following that, we derive at the first order partial derivatives of V in terms of H as well as the partial derivatives of H with respect to t and R,

$$\begin{aligned}
\frac{\partial V}{\partial t}&=\frac{\partial}{\partial S}(SH)=H+S\frac{\partial H}{\partial S}=H-S\frac{R}{S}\frac{\partial H}{\partial R}=H-R\frac{\partial H}{\partial R},\\
\frac{\partial V}{\partial A}&=\frac{\partial}{\partial A}(SH)=S\frac{\partial H}{\partial A}=S\frac{1}{S}\frac{\partial H}{\partial R}=\frac{\partial H}{\partial R}.
\end{aligned}$$

Next, we also derive the second derivative of V with respect to S,

$$\frac{\partial^2 V}{\partial S^2}=\frac{\partial}{\partial S}\left(H+S\frac{\partial H}{\partial S}\right)=2\frac{\partial H}{\partial S}+S\frac{\partial^2 H}{\partial S^2}=-\frac{2R}{S}\frac{\partial H}{\partial R}+\frac{2R}{S}\frac{\partial H}{\partial R}+\frac{R^2}{S}\frac{\partial^2 H}{\partial^2 R^2}=\frac{R^2}{S}\frac{\partial^2 H}{\partial R^2}.$$

Substituting this into Equation (3) yields the following:

$$\begin{aligned}
S\frac{\partial H}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{R^2}{S}\frac{\partial^2 H}{\partial R^2}+rSH-rSR\frac{\partial H}{\partial R}+\frac{S-A}{t}\frac{\partial H}{\partial R}-rSH=0,\\
S\frac{\partial H}{\partial t}+\frac{1}{2}\sigma^2R^2\frac{\partial^2 H}{\partial R^2}-rSR\frac{\partial H}{\partial R}+S\frac{1-R}{t}\frac{\partial H}{\partial R}=0.
\end{aligned}$$

Then, by dividing with S, the price of the Asian option can be obtained as follows:

$$\frac{\partial H}{\partial t}+\frac{1}{2}\sigma^2R^2\frac{\partial^2 H}{\partial R^2}-rR\frac{\partial H}{\partial R}+\frac{1-R}{t}\frac{\partial H}{\partial R}=0, \tag{4}$$

where payoff function, $H(T,R)=\max(1-R,0)$.

Boundary conditions. We employ boundary conditions by following [11], when the price of underlying asset, $S\to 0$, then Equation (3) converges to

$$\frac{\partial V}{\partial \tau}=-rV,$$

and as $S\to\infty$, we have

$$\frac{\partial^2 V}{\partial S^2}\simeq 0.$$

This condition determined via the option payoff since it is a Dirichlet condition. Hence, Equation (3) can be reduced to the PDE when $S=0$ and $S\to\infty$.

While for the American-style option, the price must satisfy the condition that $V(S(\tau),\tau)\geqslant V(S(\tau),0)$ at all times τ over the option contract's lifetime.

The boundary conditions for Equation (4) are straightforward, hence the convection term at $R=0$ becomes $\frac{1}{t}\frac{\partial H}{\partial R}$. The option will not be exercise as $R\to\infty$, thus $H\to 0$.

American-Style Asian option. The price for an American-style Asian option can be determined by solving a variational inequality as given by following problem:

$$\frac{\partial H}{\partial t}+\frac{1}{2}\sigma^2R^2\frac{\partial^2 H}{\partial R^2}-rR\frac{\partial H}{\partial R}+\frac{1-R}{t}\frac{\partial H}{\partial R}\leqslant 0,$$

$$H \geqslant \Omega,$$
$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} - rR\frac{\partial H}{\partial R} + \frac{1-R}{t}\frac{\partial H}{\partial R}(H - \Omega) = 0,$$
$$H(T,R) = \Omega(R),$$

where $\Omega(T,R) = \max(1-R,0)$ on $\mathbb{R}$ and we conclude that H satisfies the PDE. Hence, these are numerical difficulties that needs to be dealt with when we are to solve this equation.

Incorporation of jumps. Consider time dt and underlying asset S follows the stochastic process under jumps diffusion environment as follows:

$$dS = \mu_t S\,dt + \sigma_t S\,dB + (q_t - 1)S\,dp, \tag{5}$$

where $\mu_t = r_t - d_t - \lambda_t k_t$ represented the drift rate, the volatility is represented by σ, the dB denoted as a GBM process, and dp stands for Poisson process. An identically independent distributed random variable, $k_t = E(q_t - 1)$, which r depends on t, indicated the expected relative jump size with $k(t)$. Here, impulse function $q_t - 1$ generates a jump from S to Sq_t. It is crucial that $dp = 1$ with probability λdt and $dp = 0$ with probability $1 - \lambda dt$, where λ is the Poisson arrival intensity, which is the expected number of event that occur per unit time.

Consider $V(t,S)$ as the option values based on the time t and the underlying asset price S. Then, determine the value of V by solving the PIDE as follows:

$$\frac{\partial V}{\partial t} + \mu_t S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda E[\Delta V] - rV = 0, \tag{6}$$

where $E[\Delta V]$ represent in the following equation

$$E[\Delta V] = E[V] - V = \int_0^\infty V(Sq)\,g(q)\,dq - V, \tag{7}$$

and the expectation operator for the given equation is represented by $E[\cdot]$. Next, we can apply the reversal time $\tau = T - t$, $\mu(\tau) = r(\tau) - \lambda(\tau)k(\tau)$, where T is the expiry time of the option and r is the continuously compounded risk-free interest rate, and $g(q)$ is the probability density function of the jump amplitude q with the properties that for all q, $g(q) \geqslant 0$:

$$g(q) = \frac{1}{\sqrt{2\pi}\rho q}\exp\left\{-\frac{(\log(q) - \nu)^2}{2\rho^2}\right\}, \tag{8}$$

where ρ^2 is the variance of the jump size probability distribution and ν is the mean. Since, the expectation operator can be represent in the following form:

$$k(\tau) = E[q(\tau) - 1] = \exp\left(\mu + \frac{\rho^2}{2}\right) - 1.$$

The PIDE given by Equation (6) can be rewritten as follows:

$$\frac{\partial V}{\partial \tau} = \mu(\tau)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \lambda\int_0^\infty V(Sq)\,g(q)\,dq - (\lambda - r)V. \tag{9}$$

Hence, Equation (4) leads to the pricing formulae for Asian option under jump-diffusion environment as provided in the following proposition.

Proposition 10. The pricing formula for an Asian option under jump-diffusion environment is given by

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} - (\lambda - r)R\frac{\partial H}{\partial R} + \frac{1-R}{t}\frac{\partial H}{\partial R} + \lambda\int_0^\infty H\,g(q)\,dq = 0. \tag{10}$$

3. Discretization

Following the approach described in [9], we transform the integral in Equation (10) into a correlation integral. Hence, the fast Fourier transform (FFT) method could be implemented to evaluate the integral term for all values of S in the jump diffusion process.

By defining the log-price of the underlying asset as $x = \log S$ and using the transformation of $y = \log q$, and $q = e^y$, we can have the following operator:

$$J(S) := \int_0^\infty V(Sq)\, g(q)\, dq = \int_{-\infty}^{\infty} H(x+z) f(z) dz =: \bar{J}(x),$$

where $f(z) = g(e^z)e^z$ and $\bar{V}(y,\tau) = V(e^y,\tau)$. Note that $f(z)$ is the probability density of a jump size of $y = \log q$. Hence, the discrete form of the correlation integral is

$$J(x_i) = \sum_{j=-\frac{M}{2}+1}^{\frac{M}{2}} \bar{V}_{i+j} f_j \Delta z, \tag{11}$$

for all feasible i, where $\bar{V}_k = \bar{V}(k\Delta x)$ and

$$f_j = \frac{1}{\Delta z}\int_{j-\frac{1}{2})\Delta z}^{(j+\frac{1}{2})\Delta z} f(z)\, dz.$$

We utilize a fully implicit approach for the standard PDE and a weighted time-stepping approach for the jump integral part to avoid algebraic complexity. The discrete equations can therefore be expressed as follows:

$$\begin{aligned} V_i^{m+1}[1 + (\alpha_i) + \beta_i + r + \alpha)\Delta\tau] - \Delta\tau\beta_i V_{i+1}^{m+1} - \Delta\tau\alpha_i V_{i-1}^{m+1} \\ = V_i^m + (1-\theta)\Delta\tau\lambda \sum_{j=-M/2+1}^{j=M/2} \xi(V^{m+1}, i, j)\bar{f}_j \Delta y + \theta\Delta\tau\lambda \sum_{j=-M/2+1}^{j=M/2} \xi(V^m, i, j)\bar{f}_j \Delta y, \end{aligned} \tag{12}$$

where the weighting variables $\theta = 0$ and $\theta = 1$ correspond to an implicit and explicit handling of the jump integral, respectively.

4. Numerical results

This section demonstrates a numerical solution to the PIDE to evaluate the American-style Asian option with the jumps-diffusion process. The penalty method presents a trade-off between early exercise and continuation values which affects the option price. However, the frequency and timing of early exercise decisions can be significantly impacted by the existence or absence of price movements in the asset. Algorithm 1 describes the implementation of the Penalty method in option pricing. According to [8], setting ρ to the tolerance TOL is an ideal choice. Hence, $\rho = \text{TOL} = 10^{-4}$ has been set in the implementation.

Algorithm 1 The Penalty Method.

1: **Function** PENALTY METHOD H, A, f, b, ρ, TOL
2: res(x) $\leftarrow f + \frac{1}{\rho}\max(b-x,0) - A_x$
3: Define matrix-valued function J with the ij:th component
4: $J_{ij}(x) \leftarrow A_{ij} + \frac{1}{\rho}$
5: **repeat**
6: $H_{\text{previous}} \leftarrow H$
7: $H \leftarrow H_{\text{previous}} + J(H_{\text{previous}})^{-1} res(H_{\text{previous}})$
8: **until** $|H - H_{\text{previous}}| < \text{TOL}$
9: **return** H

MATLAB is used to program all computations. The parameters of the numerical simulations have been applied: $r = 0.2$, $T = 1$, $\sigma = 0.3$, $K = 100$, $\lambda = 0.2$, $\theta = 0.5$, $\nu = 0.2$, $\phi = 0.5$, and $\gamma = 1$. Figure 1 shows the numerical solution of the PIDE with the penalty term to find the price of the American-style Asian options jumps diffusion process comparing to the payoff function. Notice that the option value starts above the payoff function due to the penalty factor. However, as the algorithm converges, two curves get closer. This convergence reassures that the numerical solution approaches the actual option value.

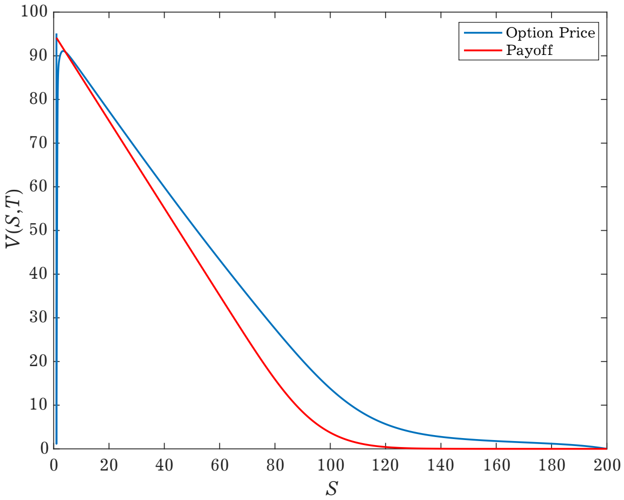

Fig. 1. Price American-Style Asian Option under Jumps Diffusion Process.

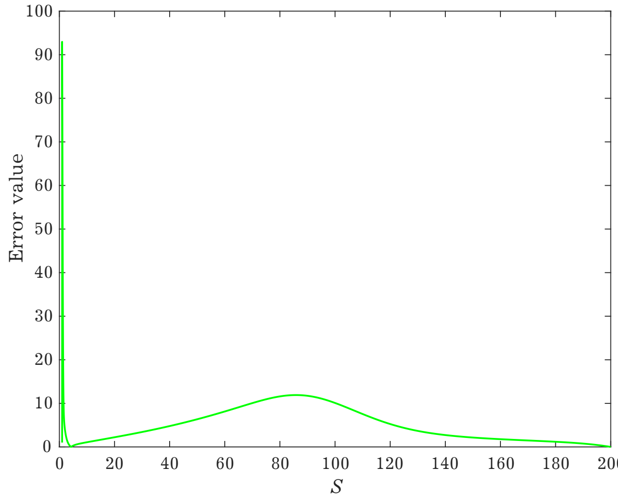

Fig. 2. Error of Evaluating the American Option with Penalty Method.

Figure 2 documents the error, which represents the error at each time step, calculated as the absolute difference between the numerical and actual options values.

5. Conclusion

In conclusion, the penalty method is valuable for handling the early exercise feature in American options. It transforms the problem of solving optimal exercise time into a mathematical optimization problem by introducing a penalty term that discourages early exercise. This approach offers a straightforward and effective means to encompass both the continuous and discontinuous components of the asset price dynamics.

To price American-style Asian options with a jump-diffusion framework, we have employed and analyzed a finite difference scheme designed to approximate the non-linear partial integro-differential equation. The evaluation of the correlation integral was accomplished by utilizing the Fast Fourier transform (FFT) method. A penalty term was seamlessly integrated into the original partial integro-differential complementarity problems to ensure compliance with the constraints inherent in the American-style option model. Numerical simulations that underscore its effectiveness have been conducted to substantiate the applicability of the penalty method in the valuation of American-style Asian options.

Acknowledgement

The authors would like to thank Universiti Putra Malaysia for the support given via Geran Putra IPS with Project Number GP-IPS/2018/9624800.

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Метод штрафу для ціноутворення в американському стилі азіатського опціону з процесом дифузії стрибків

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Опціони в американському стилі є важливими похідними контрактами на сучасних світових фінансових ринках. Вони торгують великими обсягами різних базових активів, включаючи акції, індекси, курси іноземної валюти та ф'ючерси. У цій роботі виведено та досліджено підхід штрафу для використання в ціноутворенні азіатського опціону в американському стилі за моделлю Мертона. Рівняння Блека–Шоулза містить невеликий нелінійний штрафний коефіцієнт. У цьому підході вільна та рухома межа, накладена функцією раннього виконання контракту, видаляється, щоб створити стійку область розв'язку. Включивши в моделі стрибкоподібну дифузію, вони можуть вловити особливості асиметрії та ексцесу розподілу прибутку, які часто спостерігаються в декількох активах на ринку. Ефективність схем досліджується за допомогою серії чисельних експериментів.

Ключові слова: *американський варіант; азіатський варіант; стрибкоподібно-дифузійний процес; метод штрафу.*