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Analytical approximate solutions to the nonlinear Fornberg–Whitham type equations via modified variational iteration method

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ABSTRACT

In this paper, we utilize the modified variational iteration method introduced by Abassy et al. for obtaining the analytical approximate solutions of the nonlinear Fornberg–Whitham equation and modified nonlinear Fornberg–Whitham equation. Here, we use the Maple 2021 software package for finding the analytical approximate solutions of the nonlinear Fornberg–Whitham equation and modified nonlinear Fornberg–Whitham equation and drawing their graphical representations. Moreover, here we provide a numerical comparison among the obtained analytical approximate solutions and exact solution for different particular cases. The obtained result of this paper confirm that the modified variational iteration method is more fruitful, straightforward, suitable and time consumed in repeated calculations than He's variational iteration method in case of finding analytical approximate solutions of the nonlinear Fornberg–Whitham equation and modified nonlinear Fornberg–Whitham equation.

1. Introduction

The main purpose of this paper is to obtain the more appropriate AAS of the NLFWE

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_xu_{xx},\ t > 0, \quad (1.1)$$

and the mNLFWE

$$u_t - u_{xxt} + u_x = uu_{xxx} - u^2u_x + 3u_xu_{xx},\ t > 0, \quad (1.2)$$

using a suitable analytical method. Here u denotes the fluid velocity in the x direction. The NLFWE given by Eq. (1.1) was first established by Whitham[1] for discussing the wave-breaking's qualitative behavior. This is a nonlinear dispersive wave equation and it has much attention in the study of the qualitative behavior of wave-breaking and the analysis property, see for instance[1-3] and their cited references. In 1978 Fornberg and Whitham[2] explain the NLFWE and obtained a peaked solution $u(x,\ t) = Ae^{-\frac{1}{2}\left|x-\frac{4}{3}t\right|}$ with an arbitrary constant A. Recently, a huge number of research works have been established for determining the travelling wave solutions to the NLFWE, see for instance[4-8] and their cited references. After that He et al.[9] has been derived the mNLFWE given by Eq. (1.2) by taking a small modification of the nonlinear term uu_x in Eq. (1.1) to u^2u_x. Some peakon and solitary wave solutions for the mNLFWE has been obtained based upon the bifurcation theory and the method of phase portrait analysis.

On the other hand, Ji-Huan He's VIM[10-11] absorbed much more attention in the last three decays as an auspicious analytical approach for finding the AAS s of NLDEs, see for instance.[10-17] This method was successfully implemented to solve various NLDEs such as Blasius' equation,[12] autonomous ordinary differential equations,[13] nonlinear dispersive equations,[14] Burger's and coupled Burger's equations,[16] NLFWE,[4,17] mNLFWE,[4,9] nonlinear polycrystalline solid equations,[18] regularized longwave equation[19] and coupled Schrödinger-KdV, shallow water and generalized KdV equations[20] etc. In spite of afore mentioned successful applications, the solution procedure of VIM has some disadvantages, namely, repeated computations and computations of unneeded terms, which consumes time and effort. To overcome these disadvantages of VIM, in 2007 Abassy et al.[21] established the mVIM by

List of abbreviations: NLDEs, nonlinear differential equations; NLPDE, nonlinear partial differential equation; NLFWE, nonlinear Fornberg–Whitham equation; mNLFWE, modified nonlinear Fornberg–Whitham equation; VIM, variational iteration method; mVIM, modified variational iteration method; AAS, analytical approximate solution; AASs, analytical approximate solutions.

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making some modifications of original VIM of He.[10-11] Motivated by the work of Abassy et al.,[21] in this paper we apply the Abassy *et al.'s* mVIM to obtain the more appropriate the analytical approximate solutions of the NLFWE given by Eq. (1.1) and the mNLFWE given by Eq. (1.2). The outline of this paper is as follows:

Present section provides an introduction of this paper. In Section 2, we give a brief discussion on He's VIM and mention some disadvantages of VIM according to Abassy et al.[21] Section 3 is used to describe the mVIM in the sense of Abassy et al.,[21] which will be applied in next section. Section 4 and Section 5 are devoted to obtain the more AASs of NLFWE and mNLFWE given by Eq. (1.1) and Eq. (1.2) respectively. Finally, we give a conclusion.

2. Notes on variational iteration method

In this section, we provide some introductory truths which will be needed to describe the main result of this article. First, we give a brief discussion on He's[10-11] VIM.

To illustrate this method we consider a general NLPDE in the following form:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t), \tag{2.1}$$

with the initial approximation

$$u(x,\ 0)=f(x), \tag{2.2}$$

where $L=\left(\frac{\partial}{\partial t}\right)$, $R(x,\ t)$ is a linear operator which has partial derivatives with respect to x and t, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous or heterogeneous term.

Now, from the VIM of He,[10-11] the following iteration formula has been construct:

$$U_{n+1}(x,\ t)=U_n(x,\ t)+\int_0^t\lambda\left\{LU_n(x,\tau)+\widetilde{\overbrace{RU_n(x,\tau)}}+\widetilde{\overbrace{NU_n(x,\tau)}}-g(x,\tau)\right\}d\tau, \tag{2.3}$$

here λ is the general Lagrange multiplier[22] which is used to identified the optimally via variational theory, $\overbrace{RU_n(x,\tau)}^{\sim}$ and $\overbrace{NU_n(x,\tau)}^{\sim}$ are restricted variations, that is $\delta\overbrace{RU_n}^{\sim}=0$, $\delta\overbrace{NU_n}^{\sim}=0$.

To calculate the variation with respect to U_n, we obtain the following stationary conditions:

$$\lambda^{/}(\tau)=0,\ \text{and}\ 1+\lambda(\tau)|_{\tau=t}=0. \tag{2.4}$$

Hence, the Lagrange multiplier can be determined as $\lambda=-1$ and for this value of λ, the iteration formula (2.3) has been obtained in the following form:

$$U_{n+1}(x,\ t)=U_n(x,\ t)-\int_0^t\left\{LU_n(x,\tau)+\widetilde{\overbrace{RU_n(x,\tau)}}+\widetilde{\overbrace{NU_n(x,\tau)}}-g(x,\tau)\right\}d\tau. \tag{2.5}$$

In the VIM, the second term on the right side of Eq. (2.5) is consider as the correction term which can be solved iteratively using the initial approximation $U_0(x,\ 0)=f(x)$.

In a monograph Abassy et al.[21] analyzed and discussed the above mentioned VIM very deeply and their observation is as follows:

- ➢ It is not possible to obtain a series solution by VIM as like Adomian decomposition method.[23]
- ➢ In the series solution obtained by VIM has two parts. The first part is the settled part of the approximate solution in which we can depend and the second part is the unsettled part of the approximate solution in which we cannot depend.
- ➢ In the existing VIM needs some modifications to overcome the west of time in the repeated calculations of unneeded terms, namely the unsettled part.

To overcome the above mentioned disadvantages of He's VIM, in 2007 Abassy et al.[21] introduced a mVIM which will be discussed in the next section.

3. Formulation of modified variational iteration method in the sense of Abassy et al

In this section, we will describe the mVIM in the sense of Abassy et al.[21] which will be used as the main tools of this paper.

For describing the mVIM, first we recall the general NLPDE given by Eq. (2.1) and Eq. (2.2). It is clear that general NLPDE given by Eq. (2.1) covers a large branch of applications such as soliton equations like Burgers', coupled Burgers', Schrödinger, KdV, modified KdV and many others important equations.

In this method, the same procedure as like He's VIM has been followed by using the following iteration formula:

$$U_{n+1}(x,\ t)=U_n(x,\ t)-\int_0^t\{R(U_n(x,\tau)-U_{n-1}(x,\tau))+(G_n(x,\tau)-G_{n-1}(x,\tau))\}d\tau,\ n\geq 1 \tag{3.1}$$

where

$$U_{-1}(x,t)=0,\ U_0(x,\ t)=f(x),\ \ U_1(x,\ t)=U_0(x,\ t)-\int_0^t\{R(U_0(x,\tau)-U_{-1}(x,\tau))+(G_0(x,\tau)-G_{-1}(x,\tau))-g(x,\ \tau)\}d\tau, \tag{3.2}$$

and $G_n(x,t)$ is obtained from the following formula:

$$NU_n(x,\ t)=G_n(x,\ t)+\mathrm{O}\left(t^{n+1}\right). \tag{3.3}$$

Eq. (3.1) can be solved iteratively for obtaining an AAS of the given NLPDE given by Eq. (2.1) and (2.2) that takes the following form:

$$u(x,\ t)\ \cong\ U_n(x,\ t), \tag{3.4}$$

where n is the number of iterative steps.

4. Solution of nonlinear Fornberg–Whitham equation via mVIM

In this section, we will obtain an AAS of the NLFWE given by Eq. (1.1) using mVIM and considering the following initial condition

$$u(x,\ o)=\frac{4}{3}e^{\frac{1}{2}x} \tag{4.1}$$

According to Fornberg and Whitham,[2] the exact travelling wave solution of the NLFWE given by Eq. (1.1) along with the initial condition (4.1) is as follows:

$$u(x,\ t)=\frac{4}{3}e^{\frac{1}{2}x-\frac{2}{3}t} \tag{4.2}$$

Now, using the mVIM's iteration formula given by Eq. (3.1) and Eq. (3.2) with

$$U_{-1}(x,t)=0,\ U_0(x,\ t)=\frac{4}{3}e^{\frac{1}{2}x},$$

$G_n(x,t)$ is calculated from the relatioin

$$-\{U_n(U_n)_{xxx}-U_n(U_n)_x+3(U_n)_x(U_n)_{xx}\}=G_n(x,t)+O(t^{n+1}),$$

and

$$U_1(x,\ t)=\frac{2}{3}e^{\frac{1}{2}x}(2-t),$$

we obtain the following iteration formula for the NLFWE given by Eq. (1.1):

$$\begin{aligned}U_{n+1}(x,\ t)\ =\ U_n(x,\ t)-\int_0^t[\{\{-(U_n)_{xxt}+(U_n)_x\}-\{-(U_{n-1})_{xxt}+(U_{n-1})_x\}\}\\+\{-\{U_n(U_n)_{xxx}-U_n(U_n)_x+3(U_n)_x(U_n)_{xx}\}\\+\{U_{n-1}(U_{n-1})_{xxx}-U_{n-1}(U_{n-1})_x+3(U_{n-1})_x(U_{n-1})_{xx}\}\}]d\tau.\end{aligned}\tag{4.3}$$

Here, the iteration formula (4.3) will be used to find the 2nd or above order approximation only.

Now with the help of symbolic software package Maple 2021, we obtain the following analytical approximate solution of the NLFWE given by Eq. (1.1):

$$U_0(x,\ t)=\frac{4}{3}e^{\frac{1}{2}x},$$

$$U_1(x,\ t)=\frac{2}{3}e^{\frac{1}{2}x}(2-t),$$

$$U_2(x,\ t)=\frac{1}{6}e^{\frac{1}{2}x}\left(8-5t+t^2\right),$$

$$U_3(x,\ t)=\frac{1}{72}e^{\frac{1}{2}x}\left(96-63t+18t^2-2t^3\right),$$

$$U_4(x,\ t)=\frac{1}{188}e^{\frac{1}{2}x}\left(384-255t+81t^2-14t^3+t^4\right),$$

$$U_5(x,\ t)=\frac{1}{5760}e^{\frac{1}{2}x}\left(7680-5115t+1680t^2-340t^3+40t^4-2t^5\right),$$

$$U_6(x,\ t)=\frac{1}{69120}e^{\frac{1}{2}x}\left(92160-61425t+20385t^2-4380t^3+630t^4-54t^5+2t^6\right),$$

$$U_7(x,\ t)=\frac{1}{1935360}e^{\frac{1}{2}x}\left(2580480-1720215t+572670t^2-125790t^3+19740t^4-\ 2142t^5+140t^6-4t^7\right),$$

We have seen that in the above mentioned solution obtained by mVIM all the unneeded terms have been eliminated and this solution is convergent to the exact solution given by Eq. (4.2) of NLFWE. For checking the efficiency of the mVIM, here we compare the seventh-order approximation $U_7(x,\ t)$ with the exact solution given by Eq. (4.2). In Table 4.1, we provide the list of the absolute errors of the AAS $U_7(x,\ t)$ for Eq. (1.1) at various values of x and t. From our observation it is clear that the AAS obtained by mVIM is in good agreement with the exact solution. Also, it is observed that the accuracy can be upgraded by using higher order approximation. The surface of the AAS and exact solution are shown in Fig. 4.1.

For comparison in Table 4.2, we have listed the time consumption in the calculation of various iterative steps of AASs of the NLFWE using Maple 2021 software package.

The time list of Table 4.2 shows that the mVIM saves both time and calculation. Here, we observed that the unneeded terms are absent in the obtained solution by mVIM and the computations are quicker than that of VIM. Therefore, we can comment that the mVIM is better than the VIM for finding the AAS of the NLFWE. It is also observed that the obtained AAS of NLFWE is convergent to its exact solution given by Eq. (4.2) within a few iterative steps.

Table 4.1
Absolute errors between the exact solution and the AAS $U_7(x,\ t)$ of the NLFWE.

Value of t	Error at $x=-5$	Error at $x=-2.5$	Error at $x=2.5$	Error at $x=5$
0.02	7.650×10^{-8}	2.671×10^{-7}	3.253×10^{-6}	0.00001135
0.04	1.302×10^{-7}	4.546×10^{-7}	5.536×10^{-6}	0.00001931
0.06	1.638×10^{-7}	5.721×10^{-7}	6.965×10^{-6}	0.00002431
0.08	1.803×10^{-7}	6.297×10^{-7}	7.667×10^{-6}	0.00002675
0.1	1.882×10^{-7}	6.363×10^{-7}	7.751×10^{-6}	0.00002705

5. Solution of modified nonlinear Fornberg–Whitham equation via mVIM

In this section, we will obtain an AAS of the mNLFWE given by Eq. (1.2) using mVIM and considering the following initial condition

$$u(x,0)=\frac{3}{4}\left(\sqrt{15}-5\right)sech^2\left(\left(\frac{1}{20}\sqrt{50-10\sqrt{15}}\right)x\right)\tag{5.1}$$

According to He et al.,[9] the exact solitary wave solution of the mNLFWE given by Eq. (1.2) along with the initial condition (5.1) is as follows:

$$u(x,\ t)=\frac{3}{4}\left(\sqrt{15}-5\right)sceh^2\left(\left(\frac{1}{20}\sqrt{50-10\sqrt{15}}\right)\left(x-\left(5-\sqrt{15}\right)t\right)\right).\tag{5.2}$$

Now, using the mVIM's iteration formula given by Eq. (3.1) and Eq. (3.2) with

$$U_{-1}(x,t)=0,\ U_0(x,\ t)=\frac{3}{4}\left(\sqrt{15}-5\right)sceh^2\left(\left(\frac{1}{20}\sqrt{50-10\sqrt{15}}\right)x\right),$$

$G_n(x,t)$ is calculated from the relatioin

$$-\left\{U_n(U_n)_{xxx}-(U_n)^2(U_n)_x+3(U_n)_x(U_n)_{xx}\right\}=G_n(x,t)+O(t^{n+1}),$$

and

$$\begin{aligned}U_1(x,\ t)=b\text{sech}(cx)^2+\\(2b\text{sech}(cx)^2\text{ctanh}(cx)+\ 2b^3\text{sech}(cx)^6\text{ctanh}(cx)\\-\ 6b\text{sech}(cx)^2\text{ctanh}(cx)\left(4b\text{sech}(cx)^2\text{tanh}(cx)^2-\ 2b\text{sech}(cx)^2c^2\left(1\right.\right.\\\left.\left.-\ \text{tanh}(cx)^2\right)\right)\ +\ b\text{sech}(cx)^2\left(-\ 8b\text{sech}(cx)^2c^3\text{tanh}(cx)^3\right.\\\left.+\ 16b\text{sech}(cx)^2c^3\text{tanh}(cx)\ \left(1\ -\ tanh(cx)^2\right)\right))t,\end{aligned}$$

we obtain the following iteration formula for the NLFWE given by Eq. (1.1):

$$\begin{aligned}U_{n+1}(x,\ t)=U_n(x,\ t)-\int_0^t[\{\{-(U_n)_{xxt}+(U_n)_x\}-\{\\-(U_{n-1})_{xxt}+(U_{n-1})_x\}\}+\{\\-\left\{U_n(U_n)_{xxx}-U_n^{\ 2}(U_n)_x+3(U_n)_x(U_n)_{xx}\right\}\\+\left\{U_{n-1}(U_{n-1})_{xxx}-U_{n-1}^{\ 2}(U_{n-1})_x+3(U_{n-1})_x(U_{n-1})_{xx}\right\}\}]d\tau.\end{aligned}\tag{5.3}$$

Here, the iteration formula (5.3) will be used to find the 2nd or above order approximation only.

Now with the help of symbolic software package Maple 2021, we obtain the following AAS of the mNLFWE given by Eq. (1.2):

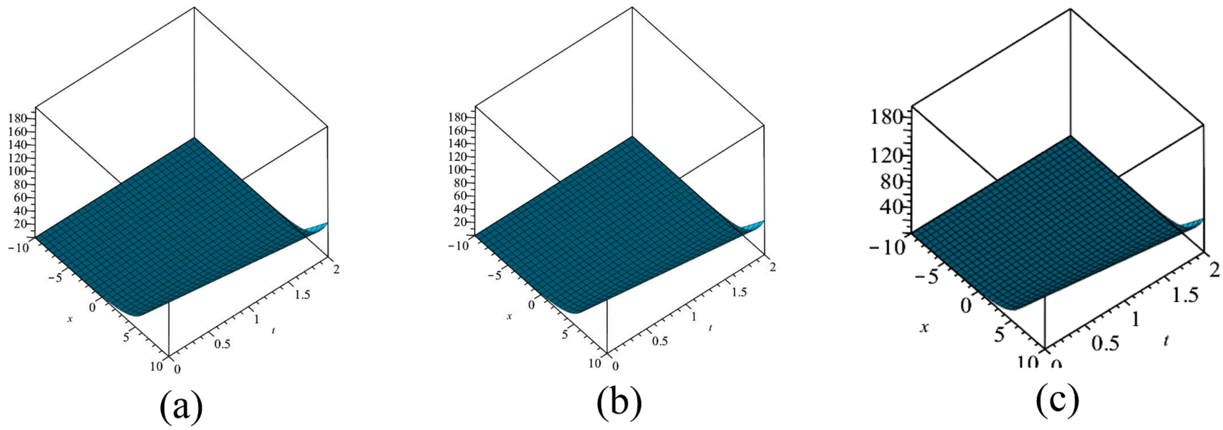

Fig. 4.1. The surfaces of (a) the exact solution of NLFWE, (b) the seventh-order AAS $U_7(x,\ t)$ of NLFWE by mVIM and (b) the seventh-order AAS $U_7(x,\ t)$ of NLFWE by VIM.

Table 4.2
The Time consumed in calculation of the AAS $U_7(x,\ t)$ of the NLFWE using Maple 2021 software package.

Order of approximation	Time consumed for the solution of NLFWE	
	mVIM	VIM
$U_1(x,\ t)$	0.60s	0.69s
$U_2(x,\ t)$	0.64s	0.64s
$U_3(x,\ t)$	0.69s	0.69s
$U_4(x,\ t)$	0.72s	0.61s
$U_5(x,\ t)$	0.67s	0.75s
$U_6(x,\ t)$	0.75s	0.78s
$U_7(x,\ t)$	0.74 s	0.82s

$$
\begin{aligned}
&b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}2^*\tanh(c^*x)\hat{}2^*(-8^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}3^*\tanh(c^*x)\hat{}3\\
&+16^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}3^*\tanh(c^*x)^*(1-\tanh(c^*x)\hat{}2))\\
&-14^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}2^*(1-\tanh(c^*x)\hat{}2)^*(-8^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}3^*\tanh(c^*x)\hat{}3\\
&+16^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}3^*\tanh(c^*x)^*(1-\tanh(c^*x)\hat{}2))\\
&-10^*b^*\mathrm{sech}(c^*x)\hat{}2^*c^*\tanh(c^*x)^*(16^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}4^*\tanh(c^*x)\hat{}4\\
&-88^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}4^*\tanh(c^*x)\hat{}2^*(1-\tanh(c^*x)\hat{}2)\\
&+16^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}4^*(1-\tanh(c^*x)\hat{}2)\hat{}2)
\end{aligned}
$$

$$
\begin{aligned}
&U_0(x,\ t)=bsech(cx)^2,\ U_1(x,\ t)\\
&=b\mathrm{sech}(cx)^2+(2b\mathrm{sech}(cx)^2c\tanh(cx)+\ 2b^3\mathrm{sech}(cx)^6c\tanh(cx)\\
&-\ 6b\mathrm{sech}(cx)^2c\tanh(cx)\left(4b\mathrm{sech}(cx)^2c^2\tanh(cx)^2-\ 2b\mathrm{sech}(cx)^2c^2\left(1\ -\tanh(cx)^2\right)\right)\\
&+\ b\mathrm{sech}(cx)^2\left(-8b\mathrm{sech}(cx)^2c^3\tanh(cx)^3+\ 16b\mathrm{sech}(cx)^2c^3\tanh(cx)\ \left(1\ -\ tanh(cx)^2\right)\right))t,U_2(x,\ t)\\
&=bsech(cx)^2+2bsech(cx)^2ctanh(cx)+2b^3sech(cx)^6ctanh(cx)\\
&-6bsech(cx)^2ctanh(cx)4bsech(cx)^2c^2tanh(cx)^2-2bsech(cx)^2c^2\left(1\ -\ tanh(cx)^2\right))\\
&+bsech(cx)^2\left(-8bsech(cx)^2c^3tanh(cx)^3+16bsech(cx)^2c^3\tanh(cx)\left(1\ -tanh(cx)^2\right)\right)\\
&+bsech(cx)^2(32bsech(cx)^2c^5tanh(cx)^5+416bsech(cx)^2c^5tanh(cx)^3\left(1\ -\ tanh(cx)^2\right)\\
&-272bsech(cx)^2c^5tanh(cx)\left(1-tanh(cx)^2\right)^2)+8bsech(cx)^2c^3tanh(cx)^3\\
&-\ 40b^3sech(cx)^6c^3tanh(cx)\left(1-tanh(cx)^2\right)\\
&-24bsech(cx)^2c^2tanh(cx)^3\left(4bsech(cx)^2c^2tanh(cx)^2-2bsech(cx)^2c^2\left(1-tanh(cx)^2\right)\right)+28
\end{aligned}
$$

$$
\begin{aligned}
&-16^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}3^*\tanh(c^*x)^*(1-\tanh(c^*x)\hat{}2)\\
&+48^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}3^*\tanh(c^*x)^*(4^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}2^*\tanh(c^*x)\hat{}2\\
&-2^*b^*\mathrm{sech}(c^*x)\hat{}2^*\hat{c}2^*(1-\tanh(c^*x)\hat{}2))^*(1-\tanh(c^*x)\hat{}2)\\
&+72^*b\hat{}3^*\mathrm{sech}(c^*x)\hat{}6^*\hat{c}3^*\tanh(c^*x)\hat{}3^*t\\
&+\left(-3^*b^*\mathrm{sech}(c^*x)\hat{}2^*c^*\tanh(c^*x)^*\left(8^*b^*\mathrm{sech}(c^*x)^2c^3\tanh(c^*x)^3\right.\right.\\
&-16^*b^*\mathrm{sech}(c^*x)^2c^3*\tanh(c^*x)^*\left(1-\tanh(c^*x)^2\right)+72^*b^3*\mathrm{sech}(c^*x)^*c^3*\tanh(c^*x)^3
\end{aligned}
$$

$$
\begin{aligned}
&-\ 40*b^3*sech(c*x)^6*c^3*tanh(c*x)*\left(1\ -\ tanh(c*x)^2\right)\\
&-\ 24*b*sech(c*x)^2*c^3*tanh(c*x)^3*\\
&\left(4*b*sech(c*x)^2*c^2*tanh(c*x)^2-\ 2*b*sech(c*x)^2*c^2*\left(1\ -\ tanh(c*x)^2\right)\right)\\
&+\ 48*b*sech(c*x)^2*c^3*tanh(c*x)*
\end{aligned}
$$

$$
\begin{aligned}
&*\left(4*b*sech(c*x)^2*c^2*tanh(c*x)^2-\ 2*b*sech(c*x)^2*c^2*\left(1\ -\ tanh(c*x)^2\right)\right)\\
&*\left(1\ -\ tanh(c*x)^2\right)+\ 28*b*sech(c*x)^2*c^2*tanh(c*x)^2\\
&*\left(-8*b*sech(c*x)^2*c^3*tanh(c*x)^3+\ 16*b*sech(c*x)^2*c^3*tanh(c*x)*\left(1\ -\ tanh(c*x)^2\right)\right)\\
&-\ 14*b*sech(c*x)^2*c^2*\left(1\ -\ tanh(c*x)^2\right)*\\
&\left(-8*b*sech(c*x)^2*c^3*tanh(c*x)^3+\ 16*b*sech(c*x)^2*c^3*tanh(c*x)*\left(1\ -\ tanh(c*x)^2\right)\right)\\
&-\ 10*b*sech(c*x)^2*c*tanh(c*x)\\
&*\left(16*b*sech(c*x)^2*c^4*tanh(c*x)^4-\ 88*b*sech(c*x)^2*c^4*tanh(c*x)^2*\left(1\ -\ tanh(c*x)^2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left.+\ 16*b*sech(c*x)^2*c^4*\left(1\ -\ tanh(c*x)^2\right)^2\right)\\
&+\ b*sech(c*x)^2*\left(-32*b*sech(c*x)^2*c^5*tanh(c*x)^5\right.\\
&+\ 416*b*sech(c*x)^2*c^5*tanh(c*x)^3*\left(1\ -\ tanh(c*x)^2\right)\\
&\left.\left.-\ 272*b*sech(c*x)^2*c^5*tanh(c*x)*\left(1\ -\ tanh(c*x)^2\right)^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-10^*b^*\mathrm{sech}(c^*x)2^*c^*\mathrm{tanh}(c^*x)^*(16^*b^*\mathrm{sech}(c^*x)2^*c4^*\mathrm{tanh}(c^*x)4\\
&-88^*b^*\mathrm{sech}(c^*x)2^*c4^*\mathrm{tanh}(c^*x)2^*(1-\mathrm{tanh}(c^*x)2)\\
&+16^*b^*\mathrm{sech}(c^*x)2^*c4^*(1-\mathrm{tanh}(c^*x)2)2)\\
&+b^*\mathrm{sech}(c^*x)2^*(-32^*b^*\mathrm{sech}(c^*x)2^*c5^*\mathrm{tanh}(c^*x)5\\
&+416^*b^*\mathrm{sech}(c^*x)2^*c5^*\mathrm{tanh}(c^*x)3^*(1-\mathrm{tanh}(c^*x)2)
\end{aligned}
$$

$$
\begin{aligned}
&-272^*b^*\mathrm{sech}(c^*x)2^*c5^*\mathrm{tanh}(c^*x)^*(1-\mathrm{tanh}(c^*x)2)2))\\
&-4^*b^*\mathrm{sech}(c^*x)2^*c2^*\mathrm{tanh}(c^*x)2+2^*b^*\mathrm{sech}(c^*x)2^*c2^*(1-\mathrm{tanh}(c^*x)2)\\
&-12^*b\text{^}3^*\mathrm{sech}(c^*x)6^*c2^*\mathrm{tanh}(c^*x)2+2^*b3^*\mathrm{sech}(c^*x)6^*c2^*(1-\mathrm{tanh}(c^*x)2)\\
&+12^*b^*\mathrm{sech}(c^*x)2^*c2^*\mathrm{tanh}(c^*x)2^*(4^*b^*\mathrm{sech}(c^*x)2^*c2^*\mathrm{tanh}(c^*x)2\\
&-2^*b^*\mathrm{sech}(c^*x)2^*c2^*(1-\mathrm{tanh}(c^*x)2)\\
&-6^*b^*\mathrm{sech}(c^*x)2^*c2^*(1-\mathrm{tanh}(c^*x)2)^*(4^*b^*\mathrm{sech}(c^*x)2^*c2^*\mathrm{tanh}(c^*x)2\\
&-2^*b^*\mathrm{sech}(c^*x)2^*c2^*(1-\mathrm{tanh}(c^*x)2))\\
&-8^*b^*\mathrm{sech}(c^*x)2^*c^*\mathrm{tanh}(c^*x)^*(-8^*b^*\mathrm{sech}(c^*x)2^*c3^*\mathrm{tanh}(c^*x)3\\
&+16^*b^*\mathrm{sech}(c^*x)2^*c3^*\mathrm{tanh}(c^*x)^*(1-\mathrm{tanh}(c^*x)2)
\end{aligned}
$$

$$
\begin{aligned}
&+b*\mathrm{sech}(c*x)2*(16*b*\mathrm{sech}(c*x)2*c4*\mathrm{tanh}(c*x)4\\
&-88*b*\mathrm{sech}(c*x)^2*c^4*\mathrm{tanh}(c*x)^2*\left(1\ -\mathrm{tanh}(c*x)^2\right)\\
&+\ 16*b*sech(c*x)2*c4*(1\ -\ tanh(c*x)2)2))\\
&4*b*sech(c*x)2*c2*tanh(c*x)2\\
&-\ 2*b*sech(c*x)2*c2*(1\ -\ tanh(c*x)2))\\
&+\frac{1}{2}*b*sech(c*x)2(-264*b*sech(c*x)2*c4*tanh(c*x)2*\\
&(4*b*sech(c*x)2*c2*tanh(c*x)2\ -\ 2*b*sech(c*x)2*c2*(1\ -\ tanh(c*x)2))\\
&1\ -\ tanh(c*x)2\\
&+\ 48*b*sech(c*x)2*c4*(1\ -\ tanh(c*x)2)2\\
&*(4*b*sech(c*x)2*c2*tanh(c*x)2\ -\ 2*b*sech(c*x)2*c2*(1\ -\ tanh(c*x)2))
\end{aligned}
$$

$$
\begin{aligned}
&+\ b*sech(c*x)2*(64*b*sech(c*x)2*c6*tanh(c*x)6\\
&-\ 1824*b*sech(c*x)2*c6*tanh(c*x)4*(1\ -\ tanh(c*x)2)\\
&+\ 2880*b*sech(c*x)2*c6*tanh(c*x)2*(1\ -\ tanh(c*x)2)2\\
&-\ 272*b*sech(c*x)2*c6*(1\ -\ tanh(c*x)2)3)\\
&-\ 80*b*sech(c*x)2*c3*tanh(c*x)3*(-8*b*sech(c*x)2*c3*tanh(c*x)3\\
&+\ 16*b*sech(c*x)2*c3*tanh(c*x)*(1\ -\ tanh(c*x)2))
\end{aligned}
$$

$$
\begin{aligned}
&+48*b*sech(c*x)2*c4*tanh(c*x)4*(4*b*sech(c*x)2*c2*tanh(c*x)2\\
&-\ 2*b*sech(c*x)2*c2*(1\ -\ tanh(c*x)2))\\
&+\ 48*b*sech(c*x)2*c2*tanh(c*x)2*(16*b*sech(c*x)2*c4*tanh(c*x)4\\
&-\ 88*b*sech(c*x)2*c4*tanh(c*x)2*(1\ -\ tanh(c*x)2)\\
&+\ 16*b*sech(c*x)2*c4*(1\ -\ tanh(c*x)2)2)\\
&-\ 24*b*sech(c*x)2*c2*(1\ -\ tanh(c*x)2)*(16*b*sech(c*x)2*c4*tanh(c*x)4
\end{aligned}
$$

$$
\begin{aligned}
&-\ 88*b*sech(c*x)2*c4*tanh(c*x)2*(1\ -\ tanh(c*x)2)\\
&+\ 16*b*sech(c*x)2*c4*(1\ -\ tanh(c*x)2)2)\\
&-12*b*sech(c*x)2*c*tanh(c*x)*(-32*b*sech(c*x)2*c5*tanh(c*x)5\\
&+\ 416*b*sech(c*x)2*c5*tanh(c*x)3*(1\ -\ tanh(c*x)2)\\
&-\ 272*b*sech(c*x)2*c5*tanh(c*x)*(1\ -\ tanh(c*x)2)2)\\
&+\ 160*b*sech(c*x)2*c3*tanh(c*x)*(-8*b*sech(c*x)2*c3*tanh(c*x)3\\
&+\ 16*b*sech(c*x)2*c3*tanh(c*x)*(1\ -\ tanh(c*x)2))*(1\ -\ tanh(c*x)2)
\end{aligned}
$$

Table 5.1
Absolute errors in the AAS $U_3(x, t)$ of the mNLFWE.

Value of t	Error at $x = 2.5$	Error at $x = 5$	Error at $x = 7.5$	Error at $x = 10$
0.02	0.0001179836181	0.0000212382	0.0000280433000	0.0000055268
0.04	0.0002362562690	0.0000479211	0.0000576961000	0.0000108327
0.06	0.0003546308000	0.0000801292	0.0000889451589	0.0000158909
0.08	0.0004729201091	0.0001179442	0.0001217772169	0.0000206737
0.1	0.0005909378380	0.0001614491	0.0001561771619	0.0000251540

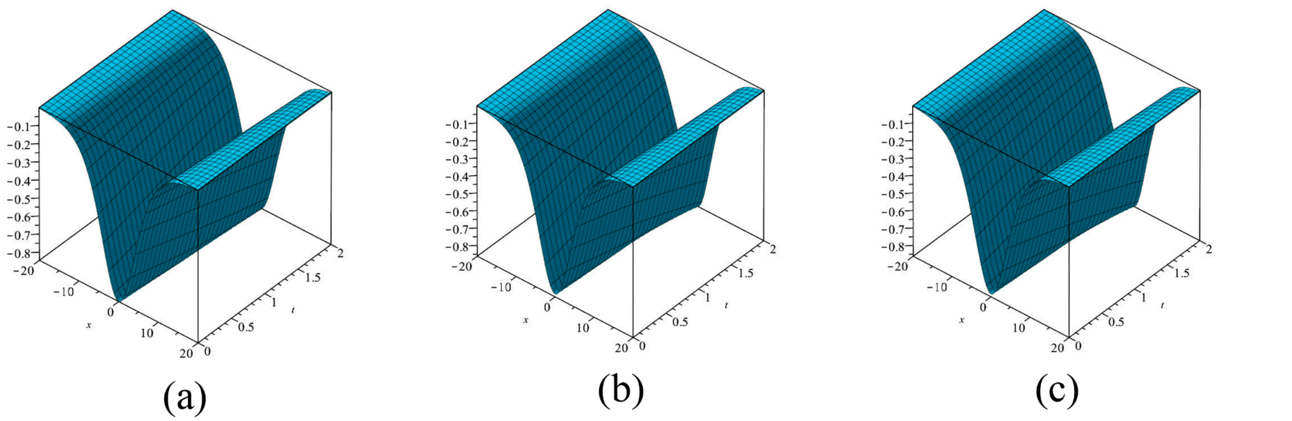

Fig. 5.1. The surfaces of (a) the exact solution of mNLFWE, (b) the seventh-order AAS $U_2(x, t)$ of mNLFWE by mVIM and (b) the seventh-order AAS $U_2(x, t)$ of mNLFWE by VIM.

Table 5.2
The Time consumed in calculation of the AAS $U_3(x, t)$ of the mNLFWE using Maple 2021 software package.

Order of approximation	Time consumed for the solution of mNLFWE	
	mVIM	VIM
$U_1(x, t)$	0.47s	0.48s
$U_2(x, t)$	0.62s	0.72s
$U_3(x, t)$	24.27s	31.36

$$+88*b*sech(c*x)^2*c^4*tanh(c*x)^2*(1-tanh(c*x)^2)$$
$$-16*b*sech(c*x)^2*c^4*(1-tanh(c*x)^2)^2$$
$$+536*b^3*sech(c*x)^6*c^4*tanh(c*x)^2*(1-tanh(c*x)^2)$$
$$-40*b^3*sech(c*x)^6*c^4*(1-tanh(c*x)^2)^2$$

$$-16*b*sech(c*x)^2*c^4*tanh(c*x)^4$$
$$-432*b^3*sech(c*x)^6*c^4*tanh(c*x)^4))$$
$$+\frac{1}{2}*((2*b*sech(c*x)^2*c*tanh(c*x)+2*b^3*sech(c*x)^6*c*tanh(c*x)$$
$$-6*b*sech(c*x)^2*c*tanh(c*x)*(4*b*sech(c*x)^2*c^2*tanh(c*x)^2$$
$$-2*b*sech(c*x)^2*c^2*(1-tanh(c*x)^2))+$$
$$b*sech(c*x)^2*(-8*b*sech(c*x)^2*c^3*tanh(c*x)^3$$

$$+16*b*sech(c*x)^2*c^3*tanh(c*x)*(1-tanh(c*x)^2)))*$$
$$(-8*b*sech(c*x)^2*c^3*tanh(c*x)^3$$
$$+16*b*sech(c*x)^2*c^3*tanh(c*x)*(1-tanh(c*x)^2)))/2$$
$$-b*sech(c*x)^2*(16*b*sech(c*x)^2*c^4*tanh(c*x)^4$$
$$-88*b*sech(c*x)^2*c^4*tanh(c*x)^2*(1-tanh(c*x)^2)$$
$$+16*b*sech(c*x)^2*c^4*(1-tanh(c*x)^2)^2)/2$$

$$-b*sech(c*x)^2*c^2*(1-tanh(c*x)^2)-b^3*sech(c*x)^6*c^2*$$
$$(1-tanh(c*x)^2)+4*b*sech(c*x)^2*c*tanh(c*x)*$$
$$(-8*b*sech(c*x)^2*c^3*tanh(c*x)^3$$
$$+16*b*sech(c*x)^2*c^3*tanh(c*x)*(1-tanh(c*x)^2))$$
$$-6*b*sech(c*x)^2*c^2*tanh(c*x)^2*(4*b*sech(c*x)^2*c^2*tanh(c*x)^2$$
$$-2*b*sech(c*x)^2*c^2*(1-tanh(c*x)^2))+3*b*sech(c*x)^2*c^2*$$
$$(1-tanh(c*x)^2)*(4*b*sech(c*x)^2*c^2*tanh(c*x)^2$$

$$
\begin{aligned}
&-2*b*sech(c*\widehat{x)2}*\widehat{c2}*(1\ -\ tanh(c*\widehat{x)2})\Big)-\frac{1}{2}*\Big(b^2*sech(c*x)^4\\
&\big(-4*b*sech(c*x)^2*c^2*tanh(c*x)^2+\ 2*b*sech(c*x)^2*c^2*\big(1\ -\ tanh(c*x)^2\big)\\
&-12*b^3*sech(c*x)^6*c^2*tanh(c*x)^2+\ 2*b^3*sech(c*x)^6*c^2*\big(1\ -\ tanh(c*x)^2\big)\\
&+12*b*sech(c*x)^2*c^2*tanh(c*x)^2\\
&*\big(4*b*sech(c*x)^2*c^2*tanh(c*x)^2-\ 2*b*sech(c*x)^2*c^2*\big(1\ -\ tanh(c*x)^2\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
&-6*b*sech(c*x)^2*c^2*\big(1\ -\ tanh(c*x)^2\big)*\\
&\big(4*b*sech(c*x)^2*c^2*tanh(c*x)^2-\ 2*b*sech(c*x)^2*c^2*\big(1\ -\ tanh(c*x)^2\big)\big)\\
&-8*b*sech(c*x)^2*c*tanh(c*x)*\\
&\big(-8*b*sech(c*x)^2*c^3*tanh(c*x)^3+\ 16*b*sech(c*x)^2*c^3*tanh(c*x)*\big(1\ -\ tanh(c*x)^2\big)\big)\\
&+b*sech(c*x)^2*\big(16*b*sech(c*x)^2*c^4*tanh(c*x)^4-\ 88*b*sech(c*x)^2*c^4*tanh(c*x)^2*\\
&\Big(\big(1\ -\ tanh(c*x)^2\big)+\ 16*b*sech(c*x)^2*c^4*\big(1\ -\ tanh(c*x)^2\big)^2\Big)\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2*\widehat{b2}*sech(c*\widehat{x)4}*(2*b*sech(c*\widehat{x)2}*c*\tanh(c*x)+2*\widehat{b3}*sech(c\\
&\quad *\widehat{x)6}*c*\tanh(c*x)-6*b*sech(c*\widehat{x)2}*c*\tanh(c*x)*(4*b*\\
&\qquad sech(c*x)^2*c^2*\tanh(c*x)^2-2*b*sech(c*x)^2*c^2\\
&\quad *(1-\tanh(c*\widehat{x)2}))+b*sech(c*\widehat{x)2}*\big(-8*b*sech(c*x)^2*c^3\\
&\qquad *\tanh(c*x)^3+16*b*sech(c*x)^2*c^3*\tanh(c*x)\\
&\quad *(1-\tanh(c*\widehat{x)2})))c*\tanh(c*x)+6*\widehat{b3}*sech(c*\widehat{x)6}*\widehat{c2}\\
&\quad *\tanh(c*\widehat{x)2}+2*b*sech(c*\widehat{x)2}*\widehat{c2}*\tanh(c*\widehat{x)2})*\widehat{t2}\\
&\quad \vdots\\
&\quad \vdots
\end{aligned}
$$

For obtain the real AASs of mNLFWE, we will put $b=\frac{3}{4}(\sqrt{15}-5)$ and $c=\left(\frac{1}{20}\sqrt{50-10\sqrt{15}}\right)$ in the above mentioned solutions. Similarly, one can obtain the rest of the AASs of mNLFWE by using Maple 2021 software package.

We have seen that in the above mentioned solution obtained by mVIM all the unneeded terms have been eliminated and this solution is convergent to the exact solution given by Eq. (5.2) of mNLFWE. Now, we show the numerical results in Table 5.1, by comparing the second-order approximation $U_2(x,\ t)$ with the exact solitary wave solution of mNLFWE given by Eq. (5.2).

For checking the efficiency of the mVIM, here we compare the second-order approximation $U_2(x,\ t)$ with the exact solution given by Eq. (5.2). In Table 5.1, we provide the list of the absolute errors of the analytical approximate solution $U_2(x,\ t)$ for Eq. (1.2) at various values of x and t. From our observation it is clear that the AAS obtained by mVIM is in good agreement with the exact solution. Also, it is observed that the accuracy can be upgraded by using higher order approximation. The surface of the AAS and exact solution are shown in Fig. 5.1.

Just for comparison in Table 5.2, we have listed the time consumption in the calculation of various iterative steps of AASs of the mNLFWE using Maple 2021 software package.

The time list of Table 5.2 shows that the mVIM saves both time and calculation. Here, we observed that the unneeded terms are absent in the obtained solution by mVIM and the computations are quicker than that of VIM. Therefore, we can comment that the mVIM is better than the VIM for finding the AASs of the mNLFWE. Moreover, it is observed that the obtained AAS of mNLFWE is convergent to its exact solution given by Eq. (5.2) with in a few iterative steps.

6. Conclusion

In this study, the mVIM in the sense of Abassy et al.[21] has been fruitfully applied for obtaining the AASs of the NLFWE and mNLFWE. The AASs given by mVIM is very much closed to the corresponding exact solution and it is expressible in a closed form. The obtained solutions has been compared with the associated exact solutions graphically and numerically and it is clear that the accurateness of the attained solutions are in a more acceptable position. The outcomes of this paper proved that the mVIM is a suitable, straightforward, powerful and time consumed mathematical tool for solving NLFWE and mNLFWE. Finally, in this study we used the symbolic software package Maple 2021 for calculating the AASs of NLFWE and mNLFWE and drawing their graphical representations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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