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SPECTRAL VARIATION OF NORMALIZED LAPLACIAN FOR VARIOUS DIRECTED AND UNDIRECTED NETWORK MODELS

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By

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In recent years, complex network research which is multidisciplinary by nature is very popular as it is a valuable tool for analysing complex real-world systems. These systems are usually studied in a large-scale data structure where they show different kinds of non-trivial topological structural properties. Since real-world systems are sometimes too large to describe explicitly, various network models have been developed to mimic their construction processes. While most of the tools used to study their structural properties are coming from graph theory, spectral analysis is another method that can be used to reveal the structural inheritance properties of a network.

In this dissertation, we focus on the studies for normalised Laplacian spectrum on six different undirected and directed network models namely, Erdös-Rényi (ER), Watts-Strogatz (WS), Barabási-Albert (BA), square grid (S_{grid}), triangular grid ($\mathrm{T}_{\mathrm{grid}}$) and growing geometrical network (GGN) network models. These network models are manipulated and constructed using Mathematica software. Spectral graph theory is used to study the network properties by utilising eigenvalues associated with the normalised Laplacian matrix computed via eigendecomposition method. Spectral measures such as spectral density plot, Cheeger constant, discrepancy and energy have been performed to analyse the directed and undirected network models.

The spectral plot for the undirected and directed networks showed very different patterns. Most of the directed network models showed a very sharp peak at eigenvalue 1 while for the undirected networks only ER, BA and S_{grid} show this feature. Undirected T_{grid} plots have two peaks, one is at 1 and another is in between {1, 1.5} while GGN has a sharp peak at 1.5 and followed by a smaller peak between {1, 1.5}. These patterns are usually unique and depend on the

network itself. Network motifs have been utilized to analyse the topological structure of these networks. It is found that ER network is mainly formed by tree motif and bipartite motif, WS network model consists of motifs with cliques, BA network model has bow-tie motifs, S_{grid} network model has square motif and finally, T_{grid} and GGN network model has triangle motif. For directed networks, the eigenvalues depend on whether the motif is cyclic or noncyclic and whether they are open or closed loops. A high number of eigenvalue 1 are produced if the motif is open or noncyclic closed loop.

From Cheeger constant (h_G) measurement, it is found that ER models has the highest h_G follow by BA and WS. This implies that ER models is difficult to be separated due the high volume of edges and random distribution of its edges. For BA and WS, h_G do not change much while T_{grid} and GGN recorded a decrease in h_G. As for S_{grid}, the h_G decreased then increased. For the directed network, the results are similar because the calculation does not take into consideration of the direction of the edges. Discrepancy of the graph for the undirected networks is also calculated using results from the Cheeger constant measurement. Results show that edges in ER models are distributed randomly while edges in S_{grid}, T_{grid} and GGN models are distributed in a controlled fashion.

In the energy measurement, adjacency matrix (A(G)) and normalised Lapalcian matrix (NL(G)) are used in both undirected and directed network models. The adjacency energy ratio (R_{AM}) of ER model is more than one implying that it is a highly random strongly regular graph with edges multiple time higher than the number of nodes. As for other networks, their R_{AM} and normalised Laplacian energy ratio (R_{NL}) values do not change much as the network size increase mainly because the edges increase linearly with the numbers of nodes. In the directed network energy measurement, energy is much smaller as compared to the undirected part because it depends on close-loop network. Open-loop and acyclic networks give zero energy value to the system.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

VARIASI SPEKTRUM NORMALISASI *LAPLACIAN* UNTUK PELBAGAI MODEL RANGKAIAN TERARAH DAN TIDAK TERARAH

Oleh

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Dalam beberapa tahun kebelakangan ini, penyelidikan rangkaian kompleks yang bersifat multidisiplin sangat popular kerana ia merupakan alat yang berharga untuk menganalisis sistem dunia sebenar yang kompleks. Sistem ini biasanya dikaji dalam struktur data berskala besar di mana ia menunjukkan pelbagai jenis sifat struktur topologi bukan remeh. Memandangkan sistem dunia sebenar kadangkala terlalu besar untuk diterangkan secara eksplisit, pelbagai model rangkaian telah dibangunkan untuk meniru proses pembinaannya. Walaupun kebanyakan alat yang digunakan untuk mengkaji sifat strukturnya datang daripada teori graf, analisis spektrum ialah kaedah lain yang boleh digunakan untuk mendedahkan sifat warisan struktur rangkaian.

Dalam disertasi ini, kami memberi tumpuan kepada kajian untuk spektrum *Laplacian* ternormal pada enam model rangkaian tidak terarah dan terarah yang berbeza iaitu, Erdös-Rényi (ER), Watts-Strogatz (WS), Barabási-Albert (BA), grid segi empat (S_{grid}), grid segi tiga (T_{grid}) and rangkaian berkembang geometri (GGN). Model rangkaian ini dimanipulasi dan dibina menggunakan perisian Mathematica. Teori graf berspektrum digunakan untuk mengkaji sifat rangkaian dengan menggunakan nilai eigen yang dikaitkan dengan matriks *Laplacian* ternormal yang dikira melalui kaedah penguraian eigen. Ukuran spektrum seperti plot ketumpatan spektrum, pemalar Cheeger, percanggahan dan tenaga telah dilakukan untuk menganalisis model rangkaian terarah dan tidak terarah.

Plot spektrum untuk rangkaian tidak terarah dan terarah menunjukkan corak yang sangat berbeza. Kebanyakan model rangkaian terarah menunjukkan puncak yang sangat tajam pada nilai eigen 1 manakala untuk rangkaian tidak terarah hanya ER, BA dan S_{grid} yang menunjukkan ciri ini. Plot T_{grid} tidak terarah mempunyai dua puncak, satu berada pada 1 dan satu lagi berada di antara {1, 1.5} manakala GGN mempunyai puncak tajam pada 1.5 dan diikuti dengan

puncak yang lebih kecil antara {1, 1.5}. Corak ini biasanya unik dan bergantung pada rangkaian itu sendiri. Motif rangkaian telah digunakan untuk menganalisis struktur topologi rangkaian ini. Didapati rangkaian ER terutamanya dibentuk oleh motif pokok dan motif bipartit, model rangkaian WS terdiri daripada motif dengan kumpulan, model rangkaian BA mempunyai motif *bow-tie*, model rangkaian S_{grid} mempunyai motif segi empat sama dan akhir sekali, model rangkaian T_{grid} dan GGN mempunyai motif segi tiga. Untuk rangkaian terarah, nilai eigen bergantung pada sama ada motif adalah kitaran atau bukan kitaran dan sama ada ia adalah gelung terbuka atau tertutup. Bilangan nilai eigen 1 yang tinggi dihasilkan jika motif terbuka atau gelung tertutup bukan kitaran.

Daripada pengukuran pemalar Cheeger (h_G), didapati model ER mempunyai h_G tertinggi diikuti oleh BA dan WS. Ini menunjukkan bahawa model ER sukar untuk dipisahkan kerana volum tepi yang tinggi dan taburan rawak garisnya. Untuk BA dan WS, h_G tidak banyak berubah manakala T_{grid} dan GGN mencatatkan penurunan dalam h_G. Bagi S_{grid}, h_G menurun kemudian meningkat. Untuk rangkaian terarah, keputusan adalah serupa kerana pengiraan tidak mengambil kira arah. Perbezaan graf untuk rangkaian tidak terarah juga dikira menggunakan hasil daripada pengukuran pemalar Cheeger. Keputusan menunjukkan bahawa tepi dalam model ER diedarkan secara rawak manakala garis dalam model S_{grid}, T_{grid} dan GGN diedarkan dalam cara terkawal.

Dalam pengukuran tenaga, A(G) dan NL(G) digunakan dalam kedua-dua model rangkaian tidak terarah dan terarah. R_{AM} model ER adalah lebih daripada satu yang membayangkan bahawa ia adalah graf sangat rawak sangat teratur dengan tepi berbilang masa lebih tinggi daripada bilangan nod. Bagi rangkaian lain, nilai R_{AM} dan R_{NL} mereka tidak banyak berubah kerana saiz rangkaian meningkat terutamanya kerana garis meningkat secara linear dengan bilangan nod. Dalam pengukuran tenaga rangkaian terarah, tenaga adalah jauh lebih kecil berbanding dengan bahagian tidak terarah kerana ia bergantung pada rangkaian gelung rapat. Rangkaian gelung terbuka dan bukan kitaran memberikan nilai tenaga sifar kepada sistem.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Declaration by Members of the Supervisory Committee

This is to confirm that:

- the research and the writing of this thesis were done under our supervision;
- supervisory responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2015-2016) are adhered to.

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A(G)	Adjacency matrix
BA	Barabási-Albert
D	Diagonal matrix
d_n	Vertex degrees
$disc(G)$	Discrepancy
E	Edges in set E
E'	Edges in set E'
E(G)	Energy
$E(\bar{G})$	Energy of the digraph
$E_A(G)$	Energy of the adjacency matrix
$E_A(\bar{G})$	Energy of the adjacency matrix for the directed graph
$E_{NL}(G)$	Energy of the normalized Laplacian matrix
$E_{NL}(\bar{G})$	Energy of the normalized Laplacian matrix for the directed graph
$e(S,\bar{S})$	Actual edges
ER	Erdös-Rényi
fMRI	Functional Magnetic Resonance Imaging
G	Graph
G'	Small graph
GGN	Growing Geometrical Network
h	Height
h_G/ h_G	Cheeger constant
$h_G(S)$	Cheeger constants of the subsets
I	Identity matrix
IRSN	Inhomogeneous Random Social Network
K_n	Complete network
$\lambda_i(A)$	Eigenvalues for adjacency matrix

$\lambda_i(NL)$	Eigenvalues for normalised Laplacian matrix
λ_n	Eigenvalues
k	First nearest neighbours
l	Number of new edges
L(G)	Laplacian matrix
m	Edges/ Links
m_1	Multiplicity of eigenvalue 1
n	Order of the network
n_i	Number of iterations
N	Vertices/ Nodes
NL(G)	Normalised Laplacian matrix
NLSDP	Normalised Laplacian spectral density plot
p	Probability
R_{AM}	Adjacency energy ratio
R_{NL}	Normalised Laplacian energy ratio
Re λ_i	Real part of eigenvalues λ_i
S_{grid}	Square grid
T_{grid}	Triangular grid
UPM	Universiti Putra Malaysia
V	Nodes in set V
V'	Nodes in set V'
w	Width
WS	Watts-Strogatz

CHAPTER 1

INTRODUCTION

In Chapter 1, we introduce the definition of network and briefly explain its importance. Then, we go through the historical background of the network theory. This is followed by introducing the network science, problem statements and research objectives. Lastly, we present the organisation of the thesis. In this thesis, “node” is interchangeable with “vertex” and “graph” is interchangeable with “network”.

1.1 Network and Its Historical Background

A network is an abstract representation of a collection of points that are linked together by lines due to certain relationships. These points are usually called nodes (or vertices) and their connection is referred to as edges (or links) as shown in Figure 1.1. It can also be called a graph since networks come from graph theory. Hence, the words “network” and “graph” are used interchangeably in this research. There are two significant types of networks as shown in Figure 1.1 namely undirected and directed networks. The connected node pairs in the undirected network are mutually affecting each other. Meanwhile, the directed edges in a directed network show the direction of the node affecting another node. Directed networks are not symmetric, and the directed link is reciprocated. The network can be further supplemented by adding weights on the edges.

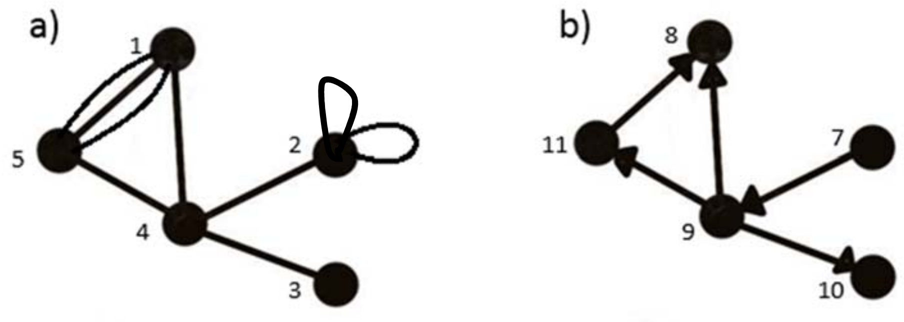

Figure 1.1 : Representation of a) undirected network and b) directed network.

Although network structures seem to be very simple, they actually possesses a lot of information if they are constructed correctly or in a meaningful way. Many complex systems from the real world such as the Internet (Newman, 2010), power grids (Amaral et al., 2000) and telephone network (Newman, 2010) can be thought of as networks. In network form, these complex systems are usually reduced to their basic connection patterns showing only nodes and edges. Simply by studying their pattern of connections using the extensive set of tools or measures in network science, important behaviour of the complex system can be revealed. Take the Internet as an example. The pattern of connections between computers on the internet affects the routes that data take over the network as well as the efficiency with which network transports those data (Newman, 2010).

Networks have a very long history in mathematics. The earliest graph theory problem was introduced by the Swiss mathematician Leonhard Euler, investigating Königsberg's bridges (now the city is called Kaliningrad) back in 1735 (Euler, 1953). The graph's history started from the first theorem of Leonhard Euler's solution that solves the famous seven bridges of Königsberg problem. This problem asks the question of whether we can cross all seven bridges without double-crossing the same bridge.

Leonhard Euler solved this question without using the geometry of the bridges. He simply uses points and lines to represent the seven bridges problem (Estrada, 2011) as shown in Figure 1.2. He found that it is only possible to cross all the bridges only once if and only if the number of bridges is even. This is because going in and out of the land would require a minimum of two bridges (even number) but in Konigsberg, there are an odd number of bridges. Hence, it is impossible to traverse the bridges without double-crossing. The solution to this problem has great significance as it is considered to be the first theorem of graph theory (Newman, 2003).

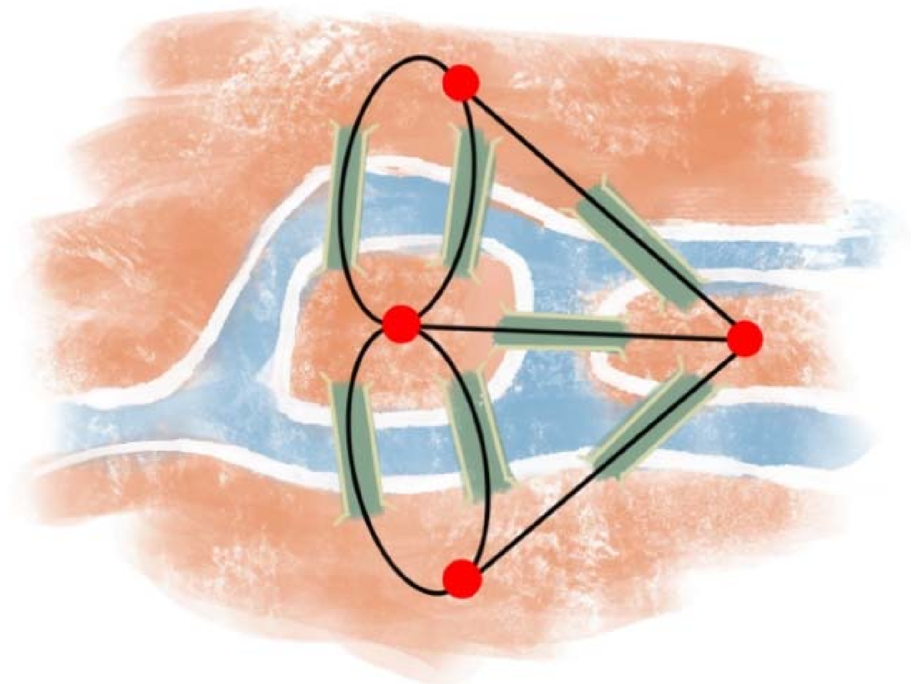

Figure 1.2 : The illustration of the seven bridges in Königsberg.

1.2 Network Science

Graph theory has played an important role in studying and analysing network structure. As increasing methodologies, tools and parameter are introduced to study networks, it eventually emerged as a new field called network science which studies complex networks that are derived from the complex real-world system. Complex networks appear in a wide range of fields namely in physics (Li, 2007), biology (Zhang, 2013), technology (Liang et al., 2019) and social science (Treur, 2012). Social network, metabolism network, Internet and World Wide Web are some examples of real-world complex networks. These networks consist of connection of nodes that represent elements of the networks such as individuals, organizations or computers and their relationship (Easley and Kleinberg, 2010).

By studying the pattern of interaction between elements as well as its topological structure, researchers have managed to gain extra information such as the overall operation or performance of the network and how the topology affects the spread of diseases, information or even rumours (Turenne, 2018). Besides, generally, complex network also can be characterised by whether it is a random or growing type of network based on its structures. Some common structural features of these networks focus primarily on properties such as the clustering coefficient of a particular node or the entire network, average path length (Smith, 2007) and network skewed degree distribution (Chattopadhyay et al., 2020).

Traditionally, the study of networks is mainly focused on regular graphs (Albert and Barabasi, 2001). As a system now have becomes more complex or too large (involving thousands to millions of nodes) as in most cases of the real-world network, many models have been proposed by researchers to mimic them. One of the earliest network models was the random network model introduced by Erdös-Rényi in 1959. This model generates a network with randomly distributed (ER) edges by connecting every pair of nodes with a certain probability (Erdös and Rényi, 1959). This model has been the cornerstone for many scientific discoveries and notable results (Castro and Grossman, 1999). Other important and notable models are Watts and Strogatz's small-world model (WS) (Watts and Strogatz, 1998), Barabási and Albert's scale-free model (BA) (Barabasi and Albert, 1999), hierarchical network model (Wang et al., 2017), deterministic model (Lachor et al., 2011), hyperbolic model (Cannon et al., 1998) and growing geometrical network model (Solé and Valverde, 2004).

1.3 Problem Statement

In recent years, complex network research, which is multidisciplinary by nature, has received a tremendous amount of attention. The main reason for its popularity is because it is a valuable tool in analysing complex real-world systems. An excellent example of the application of network theory is the strategic planning to fight COVID-19. Researchers invented a network for managing the COVID-19 pandemic and sustaining the economy (Nishi et al., 2020). Based on the study, most COVID-19 infection curves exhibit linear growth and because of the effect of nonpharmaceutical interventions (NPIs) like national lockdown, the infection curves tend to bend and level off (Thurner et al., 2020). Hence, complex network research was important because complex networks helped to track COVID-19 cases by using the NPIs. For another example of network studies in Malaysia, the network is formed by Covid-19's susceptible, infected or recovered individuals where the nodes represent the individuals and the edges represent the friendship between the individuals. The target individuals are students and this network is built to model the practical reopening strategies in an institutional setting (Fatimah and Paul, 2021).

A complex system is usually studied in a large-scale data structure. Many systems show some non-trivial topological structural properties, namely small world phenomenon, power-law degree distribution, community structures, and different motifs. These structural properties depend highly on the internodes interaction. Any changes to their connectivity will affect the properties and construction of the system. The interaction may follow certain dynamics and as a result, the system will adopt a certain inheritance structure (Banerjee, 2008). Conversely, if anything happens to the network structure, this will affect the dynamics properties as well. This interplay between the dynamics and the structures makes it worthwhile to analyse the structure and inheritance properties.

As mentioned in Section 1.2, real-world networks are too large to describe explicitly because direct visualisation only applicable when the network is sparse or only involves a small number of nodes. It will be too complicated for the eye to comprehend if the network goes up to a few thousand nodes. Various network models such as ER, WS and BA network models have been considered to study this problem. These models will attempt to mimic the real-world network construction processes to reproduce similar inheritance structural properties. Various parameters, such as degree distribution, average path length, diameter, betweenness centrality, transitivity or clustering coefficient (Newman, 2004), are measured from the networks to perform further studies. In addition, Banerjee (2008) has constructed many networks to analyse the structural inheritance properties of a network. Via these kinds of characterisation, it is found that networks that share common properties are usually similar in their topological structure.

Besides using parameters and tools developed from graph theory to analyse or characterise the network structure, spectral analysis is another method that can be used to reveal the structural inheritance properties of a network. Spectral graph theory is a field studying networks using a set of eigenvalues that can be decomposed from the connectivity matrices (more details will be discussed in Sections 3.2 and 3.3). According to Chung F. R. (1997), spectral graph theory can provide suitable methods for capturing the network's qualitative features, such as identifying the character of the network by looking at its network structure. By looking at the structure of the graph, the properties of the network can be identified. Thus, spectral analysis benefits people by saving time.

In this research work, we apply the spectral graph theory method to determine the inheritance structure on the directed and undirected network generated by network models such as Erdös-Rényi (ER), Watts-Strogatz (WS), Barabási-Albert (BA), square grid (S_{grid}), triangular grid (T_{grid}) and growing geometrical network (GGN). Although some of these studies have been performed using Laplacian and adjacency matrices on network models such as ER, BA and WS (Banerjee, 2008), work from normalised Laplacian is lacking. Besides, we also extend our study on the growing geometrical network constructed using the Fuchsian group (Taha et al., 2016) and the grid network, namely S_{grid} and T_{grid} (Hermanis & Nesenbergs, 2012). The spectrum of these networks is utilised to plot spectral density plots and performed spectral measures such as Cheeger constant and energy. Later, these measurements are analysed to characterise the directed and undirected network models qualitatively.

1.4 Research Objectives

The objectives of this work are:

- To construct the network models such as Erdös-Rényi (ER), Watts-Strogatz (WS), Barabási-Albert (BA), square grid (S_{grid}), triangular grid (T_{grid}) and growing geometrical network (GGN) with the directed and undirected connections.
- To compute and compare normalised Laplacian spectrum of network models with different connections using spectral density plots.
- To determine and compare Cheeger constant and energy of different network models from the normalised Laplacian spectrum.

1.5 Organisation of Thesis

Chapter 1 introduces the network, the history of the network and network science. Other than that, objectives and problem statements are stated clearly in this chapter. In Chapter 2, we briefly explain the real-world networks, network models and characteristics of networks. Next, in Chapter 3, the fundamentals and methodology used in this thesis are presented. We start with explaining the mathematical fundamentals of networks and followed by eigenvalues of the connectivity matrix. Later, we describe the spectral measurement as well as the generation of network models.

The results and discussions are shown in Chapter 4; all the visualisation, basic information and spectral analysis on network models are presented here. Lastly, Chapter 5 concluded all the studies with a summary and proposed future research at the end.

REFERENCES

Abedijaberi, A. (2018). Motif-level Robustness Analysis of Power Grids. *IEEE International Conference on Data Mining Workshops (ICDMW).*

Amaral, L. A. N., Scala, A., Barthélémy, M., & Stanley, H. E. (2000). Classes of small-world networks. *Proceedings of the National Academy of Sciences, vol. 97, no. 21, pp. 11149–11152*.

Arkin, E. M., Fekete, S. P., Islam, K., Meijer, H., Mitchell J. S. B., Núñez-R., Y., Polishchuk, V., Rappaport, D., & Xiao, H. (2009). Not being (super)thin or solid is hard: A study of grid Hamiltonicity. *Computational Geometry, vol. 42, no. 6, pp. 582–605.*

Arnaboldi, V., Passarella, A., Conti, M., & Dunbar, R. (2015). Human Social Networks. *in Online Social Networks Chapter 2, Eds. Boston: Elsevier, 2015, pp. 9–35.*

Balakrishnan, R. (2004). The energy of a graph. *Linear Algebra and its Applications, vol. 387, pp. 287–295.*

Banerjee, A. (2008). The Spectrum of the Graph Laplacian as a Tool for Analyzing Structure and Evolution of Networks. *Faculty of Mathematics and Computer Science, University of Leipzig.*

Banerjee, A., & Jost, J. (2008). On the spectrum of the normalized graph Laplacian. *Ph.D. dissertation, Linear Algebra and its Applications, p. 3015-3022.*

Barabasi, A.L. & Albert, R. (1999). Emergence of scaling in random networks. *Science 286, 509—512.*

Barrenetxea, G., Berefull-Lozano, B., & Vetterli, M. (2006). Lattice networks: capacity limits, optimal routing, and queueing behavior. *IEEE/ACM Transactions on Networking, vol. 14, no. 3, pp. 492-505.*

Berners-Lee, T., Cailliau, R., Groff, J.-F., & Pollermann, B. (2010). World-Wide Web: The Information Universe. *Internet Research. 20. 461-471.*

Bjelland, J., Canright, G., & Engø-Monsen, K. (2008). Web link analysis: Estimating a documentś importance from its context. *Telektronikk, 1*, 95-113.

Bloch, F., Jackson, M., & Tebaldi, P. (2016). Centrality Measures in Networks. *SSRN Electronic Journal*.

Butler, S. (2006). Spectral graph theory Cheeger constants and discrepancy. *Center for Combinatorics, Nankai University, Tianjin.*

Butler, S. (2008). Eigenvalues and structures of graphs. *Ph.D. dissertation, UC San Diego.*

Butler, S. (2016). Algebraic aspects of the normalized Laplacian. *Iowa State University, Ames, IA 50011, USA.*

Cameron, P. J. (2003). Strongly regular graphs. *Discrete Mathematics, vol. 273, no. 1, pp. 103–114.*

Cannon, J. W., Floyd, W. J., Kenyon, R., and Parry, W. R. (1998). Hyperbolic Geometry. *Flavors of Geometry, MSRI Publications, Volume 31, 1997.*

Castro, R., & Grossman, J. (1999). Famous trails to Paul Erdős. With a sidebar by Paul M. B. Vitanyi, *The Mathematical Intelligencer, vol. 21*.

Cavers, M., Fallat, S. & Kirkland, S. (2010). On the normalized Laplacian energy and general Randic index R_{-1} of graphs. *Linear Algebra and its Applications, 433 (1). pp. 172-190. ISSN 0024-3795.*

Chan, K. T. (2019). Extracting Network Structure for International and Malaysia Website via Random Walk. *ASM Science Journal, vol. 12, pp. 1–10*.

Chattopadhyay, S., Das, A., & Ghosh, K. (2020). Finding patterns in the degree distribution of real-world complex networks: going beyond power law. *Pattern Analysis and Applications, 23*.

Chung, F. (2007). Random walks and local cuts in graphs. *Linear Algebra and Its Applications - Linear Algebra Appl, 423*, 22-32.

Chung, F. R. (1997). Spectral Graph Theory. *Conference Board of the Mathematical Sciences.*

Crossley, N., Mechelli, A., Vértes, P., Winton-Brown, T., Patel, A., Ginestet, C., Mcguire, P., & Bullmore, E. (2013). Cognitive relevance of the community structure of the human brain functional coactivation network. *Proceedings of the National Academy of Sciences of the United States of America. 110. 10.1073/pnas.1220826110.*

Dode, A., & Hasani, S. (2017). PageRank Algorithm. *IOSR Journal of Computer Engineering.*

Easley, D. & Kleinberg, J. (2010). Networks, Crowds and Markets: Reasoning about a Highly Connected World. *Cambridge University Press, Cambridge.*

Erdos, P., & Renyi, A. (1959). On random graphs. *Publicationes Mathematicae 6, 290-297*.

Estrada, E. (2011). The Structure of Complex Networks: Theory and Applications. *USA: Oxford University Press, Inc.*

Euler, L. (1953). Leonhard Euler And The Koenigsberg Bridges. *Scientific American, INC*.

Farkas, I., Derényi, I., Barabasi, A.-L. & Vicsek, Tamás. (2001). Spectra of "Real-World" Graphs: Beyond the Semi-Circle Law. *Physical review. E, Statistical, nonlinear, and soft matter physics. 64. 026704. 10.1103/PhysRevE.64.026704.*

Fatimah, R., & Paul, E. (2021). Modelling the Spread of Covid-19 on Malaysian Contact Networks for Practical Reopening Strategies in an Institutional Setting. *Sains Malaysiana 50(5)(2021): 1497-1509.*

Fiedler, M. (1973). Algebraic Connectivity of Graphs. *Czechoslovak Mathematical Journal, 23*, 298-305.

Greenbaum, G., Hoffman, O., Shalev, O., & Zelnik, Y. (2015). Complex networks in ecology. *Israel Journal of Ecology & Evolution, vol. 61, pp. 67–68*.

Gutman, I. (2001). The Energy of a Graph: Old and New Results. *Algebraic Combinatorics and Applications Springer Verlag*.

Hammer, P. L., & Kelmans, A. K. (1996). Laplacian spectra and spanning trees of threshold graphs. *Discrete Applied Mathematics, vol. 65, no. 1, pp. 255–273.*

Hermanis, A., & Nesenbergs, K. (2012). Grid shaped accelerometer network for surface shape recognition. *2012 13th Biennial Baltic Electronics Conference, (pp. 203-206).*

Hurd T. R. (2020). COVID-19: Analytics of contagion on inhomogeneous random social networks. *Infectious Disease Modelling, 6, 75–90.*

Jahanbakht, N. (2010). Energy of graphs and digraphs. *University of Lethbridge. Faculty of Arts and Science.*

Jamakovic, A. (2008). Characterization of complex networks: Application to robustness analysis. *Next Generation Infrastructures Foundation.*

Jia, S., Gao, L., Gao, Y., & Wang, H. (2014). Anti-triangle centrality-based community detection in complex networks. *IET systems biology, 8*, 116-25.

Kitsak, M., Gallos, L., Havlin, S., Liljeros, F., Muchnik, L., Stanley, H. & Makse, H. (2010). Identification of influential spreaders in complex networks. *Nature Physics, vol. 6*.

Koolen, J., Vincent, M., Gutman, I., & Dusica, V. (2000). More hyperenergetic molecular graphs. *Journal of the Serbian Chemical Society, vol. 65.*

Lachor, Paweł & Puszynski, Krzysztof & Polanski, Andrzej. (2011). Deterministic models and stochastic simulations in multiple reaction models in systems biology. *Journal of Biotechnology Computational Biology and Bionanotechnology REVIEW PAPER. 92. 265-280. 10.5114/bta.2011.46543.*

Laszka, A., Szeszlér, D., Buttyan, & Levente. (2012). Linear Loss Function for the Network Blocking Game An Efficient Model for Measuring Network Robustness and Link Criticality. *Proceedings of the 3rd Conference on Decision and Game Theory for Security (GameSec)*, *7638*, pp. 152-170.

Li, L. (2007). Topologies of Complex Networks: Functions and Structures. *Ph.D. dissertation, California Institute of Technology Pasadena, California.*

Li-Qing, Q., Yong-Quan, L., & Zhuo-Yan, C. (2014). A Novel Algorithm for Detecting Local Community Structure Based on Hybrid Centrality. *Journal of Applied Sciences, 14*, 3532-3537.

Liang, Y.S.J., Chan, K. T., Zainuddin, H. & Shah, N. M. (2019). Extracting Network Structure for International and Malaysia Website via Random Walk. ASM Science Journal. 12. 1-10. 10.32802/asmscj.2019.400.

Lin, J., & Ban, Y,. (2013). Complex Network Topology of Transportation Systems. *Transport Reviews, vol. 33.*

Lu, Z., Wahlström, J., & Nehorai, A. (2018). Community Detection in Complex Networks via Clique Conductance. *Scientific Reports, 8*, 5982.

Mikolaj, M., & Tomasz, K. (2018). Graph Energies of Egocentric Networks and Their Correlation with Vertex Centrality Measures. *Entropy. 20. 916. 10. 3390/ e20120916.*

Milo, R. a. (2002). Network Motifs: Simple Building Blocks of Complex Networks. *Science 298, 824.*

Mohideen, B. (2017). A new approach to complete graph. *New Trends in Mathematical Science, vol. 1, pp. 264–268.*

Musial, K., & Kazienko, P. (2012). Social networks on the Internet. *World Wide Web. 16. 10.1007/s11280-011-0155-z.*

Murai, T. (2002). Spectral Analysis of Directed Complex Networks. *Master dissertation, Department of Physics, Graduate School of Science and Engineering, Aoyama Gakuin University.*

Nandi, S., Sen, S., Ghosh, S., & Das, S. (2015). On labeling of triangular lattice. *Electronic Notes in Discrete Mathematics, 48*.

Newman, M. E. J. (2003). The Structure and Function of Complex Networks. *SIAM Review, 45, 167-256.*

Newman, M. E. J. (2004). Detecting community structure in networks. *Eur. Phys. J. B 38, 321–330*.

Newman, M. E. J. (2010). Networks: an introduction. *Oxford; New York: Oxford University Press.*

Nicola, G., Cerchiello, P., & Aste, T. (2020). Information Network Modeling for U.S. *Banking Systemic Risk. Entropy, vol. 22, no. 11.*

Nishi, A., Dewey, G., Endo, A., Neman, S., Iwamoto, S. K., Ni, M. Y., & Young, S. D. (2020). Network interventions for managing the COVID-19 pandemic and sustaining economy. *Proceedings of the National Academy of Sciences, 117*, 30285-30294.

Nykamp, D. Q. (2020). Scale-free networks. *From Math Insight. http://mathinsight. org/scale_free_network.*

Oldham, S., Fulcher, B., Parkes, L., Arnatkeviciute, A., Suo, C., & Fornito, A. (2018). Consistency and differences between centrality metrics across distinct classes of networks. *Brain and Mental Health Research Hub, School of Psychological Sciences and the Monash Institute of Cognitive and Clinical Neurosciences (MICCN), Monash University.*

Orlovich, Y., Gordon, V., & Werner, F. (2006). Cyclic Properties of Triangular Grid Graphs. *IFAC Proceedings Volumes (IFAC-PapersOnline). 12.*

Pattanayak, H. S., Verma, H. K., & Sangal, A. L. (2020). Lengthening of average path length in social networks due to the effect of community structure. *Journal of King Saud University - Computer and Information Sciences*.

Pena I., & Rada, J. (2008). Energy of digraphs. *Linear and Multilinear Algebra, vol. 56, no. 5, pp. 565–579.*

Possamai, L. (2013). Multidimensional analysis of complex networks. *Ph.D. dissertation, Alma Mater Studiorum Università di Bologna.*

Rada, J. (2009). The McClelland inequality for the energy of digraphs. *Linear Algebra and its Applications, vol. 430, no. 2, pp. 800–804.*

Reppas, A. I., Spiliotis, K., & Siettos, C. I. (2015). Tuning the average path length of complex networks and its influence to the emergent dynamics of the majority-rule model. *Mathematics and Computers in Simulation, 109, 186-196.*

Ribeiro, P., Silva, F., & Kaiser, M. (2009). Strategies for Network Motifs Discovery. *2009 Fifth IEEE International Conference on e-Science, (pp. 80-87).*

Romero, J., Salazar, A., & Finke, J. (2017). Preferential attachment with power law growth in the number of new edges. *2017 IEEE 56th Annual Conference on Decision and Control (CDC), (pp. 2680-2685).*

Sarkar, C., & Jalan, S. (2018). Spectral properties of complex networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science, 28, 102101.*

Schintler, L., Reggiani, A., Kulkarni, R., & Nijkamp, P. (2003). Scale-Free Phenomena in Communication Networks: A Cross-Atlantic Comparison. *European Regional Science Association, ERSA conference papers*.

Shah, S. (2019). Analysis of E-Commerce Product Graphs. *AI Sciences, Target Corporation, Sunnyvale, CA 94086, USA.*

Sharan, R., Shabi, A., & Benyamini, T. (2012). Analysis of Biological Networks: Network Motifs PPI Networks: Data Preprocessing and Functional Annotation. *Tel Aviv University.*

Skiena, S. (1990). The Cycle Structure of Permutations 1.2.4. *In: Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica, Addison-Wesley, Reading, 20-24.*

Smith, R. (2007). Average Path Length in Complex Networks: Patterns and Predictions. *Bouchet-Franklin Research Institute, P.O. Box 10051 ,Rochester, NY 14610.*

So, M. K., Tiwari, A., Chu, A. M., Tsang, J. T., & Chan, J. N. (2020). Visualizing COVID-19 pandemic risk through network connectedness. *International Journal of Infectious Diseases, 96*, 558-561.

Solé, R.V., & Valverde, S. (2004). Information Theory of Complex Networks: On Evolution and Architectural Constraints. *Lecture Notes in Physics, 650*, 189-207.

Strogatz, S. (2001). Exploring Complex Networks. *Nature **410**, 268–276 (2001).*

Taha, M. H., Tim, C. K. & Zainuddin, H. (2016). Construction of Network Based on Modular group. *The Malaysian Solid State Science and Technology Society, 17, 93*.

Telesford, Q. K., Joyce, K. E., Hayasaka, S., Burdette, J. H., & Laurienti, P. J. (2011). The ubiquity of small-world networks. *Brain connectivity*, *1*(5), 367–375.

Thurner, S., Klimek, P., & Hanel, R. (2020). A network-based explanation of why most COVID-19 infection curves are linear. *Proceedings of the National Academy of Sciences, 117*, 22684-22689.

Tunalı, V., & Güvenoğlu, E. (2019). Complex Network Analysis Approach to Examining Undergraduate Program Preferences. *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi. 8. 176-186.*

Turenne, N. (2018). The rumour spectrum. *PloS one, 13(1), e0189080.*

Treur, J. (2012). Social Networks: an Overview. *Inbook: Behaviour Dynamics in Social Networks, 2012 Edition, Chapter 2, VU University Amsterdam.*

Tunalı, V., & Güvenoğlu, E. (2019). Complex Network Analysis Approach to Examining Undergraduate Program Preferences. *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi, vol. 8, pp. 176–186.*

Wang, C., Huang, N., Zhang, S., Zhang, Y., & Wu, W. (2017). A hierarchical network model for network topology design using genetic algorithm. *MATEC Web of Conferences. 119. 01008. 10.1051/matecconf/201711901008.*

Wang, X., Liu, G., & Li, J. (2017). Overlapping Community Detection Based on Structural Centrality in Complex Networks. *IEEE Access, 5*, 25258-25269.

Watts, D. J. (1999). Small Worlds, The Dynamics of Networks between Order and Randomness. *Princeton University Press, Princeton, New Jersey*.

Watts, D. & Strogatz, S. (1998). Collective dynamics of 'small-world' networks. *Nature 393, 440–442 (1998).*

Weisstein, E. W. (2001). Grid Graph. *From MathWorld- A Wolfram Web Resource. https://mathworld.wolfram.com/GridGraph.html*

Wernicke, S. (2005). A Faster Algorithm for Detecting Network Motifs. *Proceedings of the 5th International Conference on Algorithms in Bioinformatics (pp. 165–177). Berlin: Springer-Verlag.*

Williams, R. (2011). Biology, Methodology or Chance? The Degree Distributions of Bipartite Ecological Networks. *PloS one, 6*, e17645.

Wong, E. a. (2011). Biological network motif detection: principles and practice. *Briefings In Bioinformatics. Vol 13. No 2. 202-215.*

Wu, Z., Menichetti, G., Rahmede, C., & Bianconi, G. (2014). Emergent Complex Network Geometry. *Scientific Reports*.

Ying, L., Hong-Duo, C., Xiu-Ming, S., & Yong, R. (2008). An estimation formula for the average path length of scale-free networks. *Chinese Physics B, 17, 2327-2332.*

Zachary, W. W. (1977). An Information Flow Model for Conflict and Fission in Small Groups. *Journal of Anthropological Research, Vol. 33, No. 4, pp. 452-473.*

Zaidi, F. (2010). Analysis, Structure and Organization of Complex Networks. *Networking and Internet Architecture [cs.NI]. Université Sciences et Technologies - Bordeaux I, English.*

Zhang, J., & Luo, Y. (2017). Degree Centrality, Betweenness Centrality, and Closeness Centrality in Social Network. *2017 2nd International Conference on Modelling, Simulation and Applied Mathematics (MSAM2017).*

Zhang, W. (2013). Network Biology: Theories, Methods and Applications. *Nova Science Publishers, ISBN: 978-1-62618-942-3.*

Zhu, S., Kou, M., Lai, F., Feng, Q., & Du, G. (2021). The Connectedness of the Coronavirus Disease Pandemic in the World: A Study Based on Complex Network Analysis. *Frontiers in Physics, 8*, 642.