UNIVERSITI PUTRA MALAYSIA

CENTRAL EXTENSION OF LOW DIMENSIONAL ASSOCIATIVE AND LEIBNIZ ALGEBRAS

NURNAZHIFA BINTI AB RAHMAN

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By

NURNAZHIFA BINTI AB RAHMAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Master of Science

June 2020

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DEDICATION

To my husband, son and parents,
Ahmad Naqiuddin Mohd Tahir, Ayyash, Ab Rahman Aini & Miskiah Budarto.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

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June 2020

Chairman : **Sharifah Kartini binti Said Husain, PhD**
Institute : **Mathematical Research**

Leibniz algebras was introduced by Louis Loday due to several considerations in algebraic K-theory. The cyclic cohomology was somehow related to Lie algebra homology. Chevalley-Eilenberg chain complex that included in Lie algebras which the exterior powers of the Lie algebra was involved. There was a new complex which a noncommutative generalization was categorized in Lie algebra. It was called Leibniz homology as Loday is the founder. As the Lie algebra homology was related to the cyclic homology, the Leibniz homology somehow was related to the Hochschild homology.

The main purpose of this thesis is to apply the Skjelbred-Sund method and find a list of isomorphism classes for associative and Leibniz algebras. The method used is to find the high dimension of algebras by using the list of low dimensional algebras. It deals with one dimensional central extension of low dimensional algebras. To get the central extension of algebras, the condition $\theta^{\perp} \cap C(L) = 0$ need to be satisfied. If the condition is not satisfied, there is no central extension of one dimensional algebra which means this method is inapplicable. In addition, some extension invariants such as center, radical, coboundary, centroid, maximum commutative subalgebra, maximum abelian subalgebra and second cohomology are needed to investigate the isomorphism problem. As the results, seven isomorphism classes of three dimensional associative algebras, eight isomorphism classes of three dimensional Leibniz algebras and 13 isomorphism classes of four dimensional Leibniz algebra are provided.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PELUASAN BERPUSAT BAGI MATRA RENDAH ALJABAR KALIS SEKUTUAN DAN LEIBNIZ

Oleh

NURNAZHIFA BINTI AB RAHMAN

Jun 2020

Pengerusi : **Sharifah Kartini binti Said Husain, PhD**
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Aljabar Leibniz telah diperkenalkan oleh Louis Loday disebabkan oleh beberapa pertimbangan dalam Teori K aljabar. Kitaran kohomologi bagaimanapun berkait dengan homologi aljabar Lie. Kitaran kompleks Chevalley-Eilenberg yang termasuk dalam aljabar Lie di mana kuasa luaran aljabar Lie telah terlibat. Terdapat sebuah kompleks baru di mana pengitlakan tak tukar tertib terdapat dalam aljabar Lie. Ianya dipanggil kohomologi Leibniz dan Loday ialah pengasasnya. Sebagaimana homologi dikaitkan dengan homologi kitaran, begitu jugalah homologi Leibniz berkait dengan homologi Hochschild.

Tujuan utama tesis ini ialah untuk menggunakan kaedah Skjelbred-Sund dan mencari senarai kelas isomorfisma bagi aljabar kalis sekutuan dan Leibniz. Kaedah yang digunakan adalah untuk mendapatkan aljabar bermatra lebih tinggi dengan menggunakan senarai aljabar bermatra rendah. Ia berkaitan dengan peluasan berpusat aljabar bermatra satu. Untuk mendapatkan peluasan berpusat bagi aljabar, syarat $\theta^{\perp} \cap C(L) = 0$ perlu dipenuhi. Jika tidak, tidak wujud peluasan berpusat aljabar bermatra satu, bermaksud kaedah ini tidak boleh digunakan. Tambahan pula, beberapa peluasan tak varian seperti pusat, radikal, kosempadan, sentroid, sub aljabar tukar tertib maksima, sub aljabar abelian maksima dan homologi kedua adalah diperlukan untuk menyiasat masalah berisomorfik. Sebagai hasilnya, tujuh kelas isomorfisma bagi aljabar kalis sekutuan bermatra tiga, lapan kelas isomorfisma bagi aljabar Leibniz bermatra tiga dan 13 kelas isomorfisma bagi aljabar Leibniz bermatra empat disediakan.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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- the research conducted and the writing of this thesis was under our supervision;
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LIST OF ABBREVIATIONS

K	Field
V	Vector space
$\mathbb{N}$	Field of natural numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\theta^{\perp}$	Radical of algebra
$\cap$	Intersection
$C(A)$	Center of algebra
$Z^2(A,K)$	Cocycle of algebra
$B^2(A,K)$	Coboundary of algebra
$H^2(A,K)$	Second cohomology of algebra
ϕ	Automorphism group of algebra
n_{As}	Maximum abelian subalgebra
Com	Maximum commutative subalgebra
d	Derivation of algebra
$Der(A)$	The set of all derivations of an algebra
$\{\}$	Empty set
a_{ij}	Entry matrix where row i and column j
E_{ij}	(ij)-entry matrix
$End(A)$	Endomorphism of algebra
$M_{m\times n}$	Matrix of dimension $m \times n$
$Alg_n(K)$	Affine Algebraic Variety Over K
$GL_n(K)$	The set of n by n Invertible Matrices Over K
$L(K)$	Subvariety of Affine Algebraic Variety Over K
G_λ	Subgroup of Linear Reductive Group
λ_{ij}^k	Structural constant
$O(\lambda)$	Orbit of λ
MSC	Matrix structural constant

CHAPTER 1

INTRODUCTION

The classification of associative and Leibniz algebras are categorized in an old problem which has been studied by many researchers. The study of Leibniz algebras is raised from a generalization of Lie algebras and its applications. Lie algebras give a significant role in different areas of mathematics and physics such as theoretical physics and quantum field theory. The classification theory of Lie algebras is related to central extensions of Lie algebras. For example in the Lie algebras case, the Leibniz central extensions give a significant role in the structural theory of Leibniz algebras. In this research, nilpotent Leibniz algebras as central extensions of nilpotent Leibniz algebras of lower dimension is completed.

Let A be a nilpotent algebra, V is a vector space and θ be a cocycle in a second cohomology of algebra, $H^2(A,K)$. Then θ defines an algebra structure on $A \oplus V$, which is called the central extension of A by θ. By this computation all nilpotent algebras of dimension n are obtained. In general, the central extension is used to enlarge from algebra dimension n to algebra dimension $n+1$. As an application, the set of all three dimensional associative algebras, three dimensional Leibniz algebras and four dimensional Leibniz algebras are described. The purpose of this introductory chapter is mainly to review briefly some familiar properties of basic concepts and some crucial materials in our study. This chapter is also used to set down the conventions and notations throughout the thesis.

1.1 Basic Concepts

In this section some basic concepts regarding vector space, algebra, associative algebra, Lie algebra and Leibniz algebra are introduced.

Definition 1.1 *(Morris, 1978) A set V is called a vector space over the K or a K-space if*
(a)(i) V is closed with respect to a binary operation called addition $(+)$,
i.e. if $u,v \in V$, then $u+v \in V$.
(ii)(Commutative axiom) $u+v = v+u$ for all $u,v \in V$.
(iii)(Associative axiom) $u+(v+w) = (u+v)+w$ for all $u,v,w \in V$.
(iv) There exist an element $0 \in V$, called the zero element, such that
$u+0 = 0+u = u$ for all $u \in V$.
(v) For every $v \in V$, there exist an element $(-v) \in V$ such that
$v+(-v) = 0 = (-v)+v$.
(b)(i) For every $\alpha \in K$ and $v \in V$. An element αv called the scalar multiple of v by α, is defined and $\alpha v \in V$.

(ii) $\alpha(u+v)=\alpha u+\alpha v$*, for all* $\alpha \in K, u, v \in V$.
(iii)$(\alpha+\beta)v=\alpha v+\beta v$*, for all* $\alpha,\beta \in K, v \in V$.
(iv) $\alpha(\beta v)=(\alpha\beta)v$*, for all* $\alpha,\beta \in K, u, v \in V$.
(v) $1v=v$ *for all* $v \in V$.

As an example, let $M_{m,n}$, or simply M, denote the set of all $m \times n$ matrices over an arbitrary K. Then $M_{m,n}$ is a vector space over K with respect to the ordinary operations of addition and scalar multiplication (Lipschutz, 1991). Another example, let K be an arbitrary field. The notation K^n is frequently used to denote the set of all n-tuples of elements in K, where K^n is a vector space over K using the following operations for vector addition :

$$(a_1,a_2,\ldots,a_n)+(b_1,b_2,\ldots,b_n)=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n).$$

For scalar multiplication :

$$k(a_1,a_2,\ldots,a_n)=(ka_1,ka_2,\ldots,ka_n).$$

The zero vector, $\mathbf{0}$ in K^n is the n-tuple of zeros, $\mathbf{0}=(0,0,\ldots,0)$ and the negative of a vector is defined by $-(a_1,a_2,\ldots,a_n)=(-a_1,-a_2,\ldots,-a_n)$ (Lipschutz and Lipson, 2002).

Polynomial is often related to vector space. For example, Martin and Mizel (1996) stated that the set of all pairs of real numbers (α,β) where $\alpha=(a_1,a_2),\beta=(b_1,b_2)$ and the addition and scalar multiplication are defined as follows:

$$\alpha+\beta=(a_1+b_1,a_2+b_2),$$

$$c\alpha=(ca_1,ca_2),$$

where c is any scalar. The K together with these rules of addition and scalar multiplication is a vector space which we denote by K^2. Next, Ayres and Jaisingh (2004) gave an example, let K be any field, $V=F[x]$ be the polynomial domain in x over K, and define usual addition and scalar multiplication in $F[x]$. Then V is a vector space over K. Later on, Anderson and Feil (2015) illustrate the fact that the multiplication vectors are not included in a vector space. Let $\mathbb{Q}_n[x]$ be the set of polynomials of degree not more than n with coefficients in $\mathbb{Q}$ that is a vector space over $\mathbb{Q}$, vectors are polynomials of degrees not more than n and scalars are rational numbers. Thus $\mathbb{Q}_n[x]$ is closed under addition. Nicholson (2019) stated that given $n \geq 1$, let P_n denote the set of all polynomials of degree at most n, together with the zero polynomial. That is

$$P_n=\{a_0+a_1x+a_1x^2+\cdots+a_nx^n|a_0,a_1,a_2,\ldots,a_n \in \mathbb{R}\}.$$

Then P_n is a vector space. Indeed, sums and scalar multiples of polynomials in P_n are again in P_n, and the other vector space axioms are inherited from P_n. In particular, the zero vector and the negative of a polynomial in P_n are the same as those in P_n.

Definition 1.2 *(Marcus and Minc, 1968) If $W = \langle v_1, \ldots, v_k \rangle$ and the vectors $v_1, \ldots, v_k$ are linearly independent, then they are said to form a basis of the (finite dimensional) space W.*

For examples, let U be the vector space of all 2×3 matrices over a field K. Then the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

form a basis of U. Thus dim $U = 6$. More generally, let V be the vector space of all $m \times n$ matrices over K and let $E_{ij} \in V$ be the matrix with $ij-$entry 1 and 0 elsewhere. Then the set $\{E_{ij}\}$ is a basis for V, consequently dim $V = mn$. Another example, consider the following $n+1$ polynomials in $\mathbb{P}_n(t)$:

$$1, t-1, (t-1)^2, \ldots, (t-1)^n.$$

The degree of $(t-1)^k$ is k; hence no polynomial can be a linear combination of proceeding polynomials. Thus the polynomials are linearly independent. Futhermore, they form a basis of $\mathbb{P}_n(t)$ and dimension of $\mathbb{P}_n(t) = n+1$. (Lipschutz, 1987).

Martin and Mizel (1996) considered $B : \beta_0, \ldots, \beta_{n-1}$ be the system of polynomials $1, t, t^2, \ldots, t^{n-1} \in P^n$, the vector space of all polynomials of degree not exceeding $n-1$. Let

$$\alpha = a_0 t^{n-1} + a_1 t^{n-2} + \cdots + a_{n-1},$$

be an ordinary member of P^n. Then

$$\alpha = a_0 \beta_{n-1} + a_1 \beta_{n-2} + \cdots + a_{n-1} \beta_0,$$

so that B clearly spans P^n. B is also independent. For if

$$c_0 \beta_{n-1} + c_1 \beta_{n-2} + \cdots + c_{n-1} \beta_0 = 0,$$

then

$$c_0 t^{n-1} + c_1 t^{n-2} + \cdots + c_{n-1},$$

is a polynomial which is identically zero. Since by the fundamental theorem of algebra only the trivial polynomial vanishes identically, it follows that

$$c_0 = c_1 = \cdots = c_{n-1} = 0.$$

Therefore B is independent, an independent spanner for P^n. Thus B is a basis of P^n.

Definition 1.3 *(Marcus and Minc, 1968) Let V be a finite dimensional vector space. The dimension of V, denoted by dim V, is the number of vectors in a basis of V. If V*

consists of n vectors, then V is n-dimensional.

Anderson and Feil (2015) stated that the dimension of the vector space $\mathbb{R}^3$ over $\mathbb{R}$ is 3. Let $e_1 = (2,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. For any vector $v = (x_1,x_2,x_3) \in \mathbb{R}^3$, we have $v = x_1e_1 + x_2e_2 + x_3e_3$. So $\mathbb{R}^3$ is spanned by e_1,e_2,e_3 and if

$$c_1e_1 + c_2e_2 + c_3e_3 = 0,$$

then $c_1 = c_2 = c_3 = 0$. Therefore e_1,e_2,e_3 are linearly independent. Hence $\{e_1,e_2,e_3\}$ forms a basis of $\mathbb{R}^3$. Meanwhile the $dim(\mathbb{R}^3) = 3$ due to the standard basis consist of three elements. Let $M = \{(2,1,-2),(-2,-1,2),(4,2,-4)\}$, then $(4,2,-4) = (2,1,-2)-(-2,-1,2)$. So these three vectors are linearly dependent. Since $dim(\mathbb{R}^3) = 3$, Thus M is not a basis of $\mathbb{R}^3$. Next, consider $N = \{(2,2),(1,-1)\}$ and let $a(1,2)+b(1,-1) = (0,0)$. Then $a+b = 0$ which gives $a = 0, b = 0$. So these two vectors are linearly independent. Since $dim(\mathbb{R}^2) = 2$ thus N is a basis of $\mathbb{R}^2$.

Definition 1.4 *(Langari, 2010) A is called an algebra over K if A is a vector space over K, such that* $f : A \times A \longrightarrow A$ *and*

$$f(\alpha a+\beta b,c) = \alpha f(a,c)+\beta f(b,c);$$

$$f(a,\gamma b+\delta c) = \gamma f(a,b)+\delta f(a,c);$$

for all $a,b,c \in A$ *and* $\alpha,\beta,\gamma,\delta \in K$.

Mohamed (2014) stated that the set of all complex numbers $\mathbb{C}$ is an algebra over the set of real numbers $\mathbb{R}$ with $dim_{\mathbb{R}}(\mathbb{C}) = 2$. One set of basis is $\{a,bi\}$ where $a,b \in \mathbb{R}$.

The set of all bilinear maps $V \otimes V \to V$ form a vector space $Hom(V \otimes V,V)$ of dimensional n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $Alg_n(K) \simeq K^{n^3}$.

An n-dimensional algebra A over K can be considered as an element $\lambda(A)$ of $Alg_n(K)$ via the bilinear mapping $\lambda : A \otimes A \longrightarrow A$ defining a binary algebraic operation on A. Let $e_1,e_2,\cdots,e_n$ be a basis of the algebra A, then the table of multiplication of A is represented by point (λ_{ij}^k) of this affine space as follows:

$$\lambda(e_i,e_j) = \sum_{k=1}^{n} \lambda_{ij}^k e_k,$$

where $i,j = 1,2,\cdots,n$. λ_{ij}^k are called structural constants of A. A Leibniz algebra A on n -dimensional vector space V over a K can be written as a pair $A = (V,\lambda)$, where λ is an algebra law on V.

Definition 1.5 *(Mohamed, 2014) An action of a group G on a set X is a function* $\star : G \times X \longrightarrow X$ *that satisfies the following conditions:*

1. $e \star x = x$, *for all* $x \in X$, *where* e *is the identity element of* G.
2. $g \star (h \star x) = (gh) \star x$, *for all* $g, h \in G$ *and* $x \in X$.

If G *is an algebraic group acting on an algebraic variety* X, *then there is the additional condition that* $\star$ *is a morphism.*

Definition 1.6 *(Mohamed, 2014) Two laws* λ_1 *and* λ_2 *from* A *are said to be isomorphic if there is* $g \in GA(V)$ *such that*

$$\lambda_2(x,y) = (g \star \lambda_1)(x,y) = g^{-1}(\lambda_1(g(x), g(y)))$$

for all $x, y \in V$. *Thus we get an action of* $GA(V)$ *on* A.

Let $O(\lambda)$ be the set of laws isomorphic to λ. It is called the orbit of λ. Once a basis is fixed, we can identify the law λ with its structure constants. These constants λ_{ij}^k satisfy for associative algebra :

$$\sum_{s}^{n} \lambda_{ij}^{s} \lambda_{sk}^{t} = \sum_{s}^{n} \lambda_{jk}^{s} \lambda_{is}^{t}, \tag{1.1.1}$$

where $i, j, k, t = 1, 2, \ldots, n$ and for Leibniz algebra :

$$\sum_{l=1}^{n} (\lambda_{jk}^{l} \lambda_{il}^{m} - \lambda_{ij}^{l} \lambda_{lk}^{m} + \lambda_{ik}^{l} \lambda_{lj}^{m}) = 0, \tag{1.1.2}$$

where $i, j, k, m = 1, 2, \ldots, n$. Then A appears as an algebraic variety embedded in the linear space of bilinear mapping on V , isomorphic to K^{n^3}. Let $\lambda \in A$ and G_λ be the subgroup of $GL_n(K)$ defined by

$$G_\lambda = \{ f \in GL_n(K) | f \star \lambda = \lambda \} .$$

Let $O(\lambda)$ be the orbit of λ with respect to action of $GL_n(K)$. It is isomorphic to the homogeneous space $GL_n(K)/G_\lambda$. Within the context of this study, the following is a particular definition of orbit function.

Definition 1.7 *(Mohamed, 2014) A function* $f : A \mapsto K$ *is said to be invariant (or orbit) function if*

$$f(g \star \lambda) = f(\lambda)$$

for all $g \in GL_n(K)$ *and* $\lambda \in A$.

Description of A for low dimensional associative and Leibniz algebras have been done by solving the system (1.1.1) and (1.1.2) with respect to λ_{ij}^k.

Definition 1.8 *(Graaf, 2010) An associative algebra As is a vector space over* K *equipped with a bilinear map*

$$\cdot : As \times As \longrightarrow As$$

satisfying the associative identity

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

for all $x, y, z \in As$.

Example 1.1 *(Abdulkareem, 2014) Let* V *be an n-dimensional vector space over* K. *The set of all endormorphism,* $End(V)$ *forms a vector space is a linear transformation from* V *to* V. *The multiplication of two elements* $f, g \in End(V)$ *is defined by*

$$(f \circ g)(v) = f(g(v)), \text{for all } v \in V.$$

This product of $End(V)$ *is an associative algebra.*

Example 1.2 *Let* As *be a three-dimensional algebra with the table of multiplication as follows :*

$$As_{2,2} : e_1 e_1 = e_2$$

that satisfy the associative identity

$$(e_i e_j) e_k = e_i (e_j e_k)$$

for all $i, j, k = 1, 2$. *Then As is an associative algebra.*

The following definitions and examples can be found in Langari (2010).

Definition 1.9 *An algebra* L *over* K *is said to be a Lie algebra if the following identities hold:*

i. Anti-symmetry: $[x,y] = -[y,x]$, *for all* $x, y \in L$.

ii. Jacobi identity: $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ *for all* $x, y, z \in L$, *where* $[\cdot,\cdot]$ *denotes the multiplication in* L.

The first example of Lie algebra is any vector space V has a Lie bracket defined by $[x,y] = 0$ for all $x, y \in V$. This is the abelian Lie structure on V. In particular the K may be regarded as a one dimensional abelian Lie algebra. Other example is let $K = \mathbb{R}$. The vector product $(x,y) \longmapsto x \wedge y$ defines the structure of a Lie algebra on $\mathbb{R}^3$. We denote this Lie algebra by $\mathbb{R}^3_\wedge$. Explicitly, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then $x \wedge y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$.

Definition 1.10 *A Leibniz algebra* L *is a vector space over* K *equipped with a bilinear map*

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for all $x,y,z \in L$.

For example, let L be a two-dimensional algebra with the table of multiplication as follows :

$$L_{2,3} : [e_1,e_2] = -[e_2,e_1] = e_2$$

that satisfy the Leibniz identity

$$[e_i,[e_j,e_k]] = [[e_i,e_j],e_k] - [[e_i,e_k],e_j]$$

for $i,j,k = 1,2$. Then L is a Leibniz algebra.

Let A be an algebra, notes that $A^1 = A, A^2 = [A,A], A^3 = [A^2,A], \ldots, A^{k+1} = [A^k,A]$, for $k \in \mathbb{N}$. Then the non ascending series as

$$A^1 \supset A^2 \supset A^3 \supset \ldots \supset A^s \supset \ldots.$$

Definition 1.11 *(Ayupov and Omirov, 2001) An algebra A is said to be nilpotent, if there exist an integer $s \in \mathbb{N}$, such that $A^s = \{0\}$.*

The smallest integer s for that $A^s = 0$ is called the nilindex of A.

Let $A = span\{e_1,e_2,e_3,e_4\}$ with the following table of multiplication

$$A : e_1e_1 = e_2, e_2e_1 = e_3, e_1e_2 = e_4.$$

We have the following non ascending series $A^1 \supset A^2 \supset A^3 \supset A^4 = \{0\}$ where $A^2 = span\{e_2,e_3,e_4\}, A^3 = span\{e_3\}$ and $A^4 = span\{\}$. Therefore A is nilpotent algebra.

The algebra is non nilpotent if the non ascending series as

$$A \supset A^2 \supset A^3 \supset \ldots \supset A^s \supset \ldots$$

where $A^s \neq \{0\}$. For example, let $A = span\{e_1,e_2,e_3\}$ with the following table of multiplication

$$A : e_2e_1 = e_2, e_3e_1 = e_3.$$

We have $A^2 = span\{e_2,e_3\}, A^3 = span\{e_2,e_3\}, \ldots, A^n = span\{e_2,e_3\}$. Then we do not have the non ascending series which is $A^n = \{0\}$. Therefore A is non nilpotent algebra.

The following two examples can be found in Langari (2010). It is stated that every abelian algebra is nilpotent with a nilindex equal to two. The other example is the three dimensional algebra, g defined on a basis $\{e_1, e_2, e_3\}$ by the bracket

$$[e_1, e_2] = e_3$$

is nilpotent with a nilindex equal to three. When

$$g = span\{e_1, e_2, e_3\}, g^2 = span\{e_3\}, g^3 = \{0\},$$

then we have the non ascending series as $g^1 \supset g^2 \supset g^3 = \{0\}$. Thus g is nilpotent algebra.

Definition 1.12 *(Graaf, 2010) Let As be an assosiative algebra and V be a vector space over K. Then the bilinear maps $\theta : As \times As \longrightarrow V$ with*

$$\theta(x,(y,z)) = \theta((x,y),z)$$

for all $x, y, z \in As$ is called associative cocycle.

For example, let $As = span\{e_1, e_2, e_3\}$ with the following table of multiplication $As_{3,2} : e_1e_1 = e_3, e_2e_2 = e_3$ be an associative algebra. The algebra satisfies

$$\theta(e_i,(e_j,e_k)) = \theta((e_i,e_j),e_k)$$

for $i, j, k = 1, 2, 3$. Then we have the following associative cocycle :

$$\theta = \begin{bmatrix} \theta(e_1e_1) & \theta(e_1e_2) & 0 \\ \theta(e_2e_1) & \theta(e_2e_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Definition 1.13 *(Rakhimov et al., 2009) Let L be a Leibniz algebra over algebraically closed field K and V be a vector space over K. Then $\theta : L \times L \longrightarrow V$ be a bilinear maps satisfying the Leibniz identity*

$$\theta([x,[y,z]]) = \theta([[x,y],z]) - \theta([[x,z],y])$$

for all $x, y, z \in L$ is called Leibniz cocycle.

As an example, let $L_{3,4} = span\{e_1, e_2, e_3\}$ with the following table of multiplication $L_{3,4} : [e_1, e_2] = e_3, [e_2, e_1] = -e_3$ be a Leibniz algebra. The algebra satisfies

$$\theta([e_i,[e_j,e_k]]) = \theta([[e_i,e_j],e_k]) - \theta([[e_i,e_k],e_j])$$

for $i, j, k = 1, 2, \ldots, n$.

Note that 27 cases are listed. In this section we show only two cases while full calculations are shown in Appendix B.

1. For case $i = j = 1, k = 2$ we have

$$\theta([e_1,[e_1,e_2]]) = \theta([[e_1,e_1],e_2]) - \theta([[e_1,e_2],e_1]),$$
$$\theta([e_1,e_3]) = -\theta([e_3,e_1]).$$

2. For case $i = 3, j = 1, k = 2$ we have

$$\theta([e_3,[e_1,e_2]]) = \theta([[e_3,e_1],e_2]) - \theta([[e_3,e_2],e_1]),$$
$$\theta([e_3,e_3]) = 0.$$

After consider for 27 cases, a Leibniz cocycle is obtain as follows :

$$\theta = \begin{bmatrix} \theta(e_1e_1) & \theta(e_1e_2) & \theta(e_1e_3) \\ \theta(e_2e_1) & \theta(e_2e_2) & \theta(e_2e_3) \\ -\theta(e_1e_3) & -\theta(e_2e_3) & 0 \end{bmatrix}.$$

Definition 1.14 *(Rakhimov and Langari, 2010) An algebra L over field K is called a center of Leibniz algebra if its binary map satisfies the following properties :*

$$C(L) = \{a \in L | [a,L] = [L,a] = 0\},$$

for all $a \in L$.

Remark : The definition for center of associative algebra is similar with definition for the center of Leibniz algebra as mentioned above.

Example 1.3 *Let $L_{3,1}$ be a 3-dimensional Leibniz algebra with a basis $\{e_1,e_2,e_3\}$. The table of multiplication be given as follows :*

$$L_{3,1} : [e_1,e_1] = e_3, [e_1,e_2] = e_3, [e_2,e_1] = e_3.$$

Then,
If $a = e_1, [e_1,e_1] \neq 0; [e_1,e_2] \neq 0; [e_2,e_1] \neq 0; [e_1,e_3] = 0; [e_3,e_1] = 0.$

If $a = e_2, [e_2,e_1] \neq 0; [e_1,e_2] \neq 0; [e_2,e_2] = 0; [e_2,e_3] = 0; [e_3,e_2] = 0.$

If $a = e_3, [e_3,e_1] = 0; [e_1,e_3] = 0; [e_3,e_2] = 0; [e_2,e_3] = 0; [e_3,e_3] = 0.$

Thus, the center of Leibniz algebra is span$\{e_3\}$.

Definition 1.15 *(Graaf, 2010) Let A be an algebra over K for $\theta \in Z^2(A,V)$. The set $\theta^{\perp}$ is called radical of algebra if its binary map satisfies the following properties:*

$$\theta^{\perp} = \{a \in A | \theta(a,b) = \theta(b,a) = 0\}$$

for all $b \in A$.

Example 1.4 *Let $As_{3,1}$ be a 3-dimensional associative algebra with a basis*

$\{e_1, e_2, e_3\}$ *and let the table of multiplication be given as follows :*

$$As_{3,1} : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_1 = e_3.$$

We have associative cocycle for $A_{3,1}$ *as follows :*

$$\theta(As_{3,1}) = \begin{bmatrix} \theta(e_1e_1) & \theta(e_1e_2) & 0 \\ \theta(e_2e_1) & \theta(e_2e_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then,
If $a = e_1, \theta(e_1e_1) \neq 0; \theta(e_1e_2) \neq 0; \theta(e_2e_1) \neq 0; \theta(e_1e_3) = 0; \theta(e_3e_1) = 0.$

If $a = e_2, \theta(e_2e_1) \neq 0; \theta(e_1e_2) \neq 0; \theta(e_2e_2) \neq 0; \theta(e_2e_3) = 0; \theta(e_3e_2) = 0.$

If $a = e_3, \theta(e_3e_1) = 0; \theta(e_1e_3) = 0; \theta(e_3e_2) = 0; \theta(e_2e_3) = 0; \theta(e_3e_3) = 0.$

Thus, the radical of algebra is span$\{e_3\}$.

The following statements are stated in (Rakhimov et al., 2015). Let A be an arbitrary algebra over K. The centroid of A, $\Gamma(A)$ is defined by

$$\Gamma(A) = \{\phi \in End(A) | \phi(xy) = \phi(x)y = x\phi(y) \text{ for } \forall x, y \in A\}.$$

A derivation of an algebra A is a K-linear transformation $d : A \longrightarrow A$ satisfying

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y), \text{ for } \forall x, y \in A.$$

The set of all derivations of an algebra A is denote as $Der(A)$.

Example 1.5 *Let* $As_{3,1} = span\{e_1, e_2, e_3\}$ *with the following table of multiplication* $As_{3,1} : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_1 = e_3$ *be an algebra. The algebra satisfies*

$$\phi(e_ie_j) = \phi(e_i)e_j = e_i\phi(e_j),$$

for all $i, j = 1, 2, 3.$

Then a centroid of algebra is obtained as follows :

$$\phi(As_{3,1}) = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{11} + a_{21} \end{bmatrix}.$$

Example 1.6 *Let* $As_{3,2} = span\{e_1, e_2, e_3\}$ *with the table of multiplication*

$$As_{3,2} : e_1e_1 = e_3$$

be an algebra. The algebra satisfies

$$d(e_ie_j) = d(e_i)e_j + e_id(e_j)$$

for all $i, j = 1,2,3$.

In this example, the method is shown as below.

1. When $i = j = 1$;

$$\begin{aligned} d(e_1)e_1 + e_1d(e_1) &= d(e_1e_1), \\ (a_{11}e_1 + a_{12}e_2 + a_{13}e_3)e_1 + e_1(a_{11}e_1 + a_{12}e_2 + a_{13}e_3) &= a_{31}e_1 + a_{32}e_2 + a_{33}e_3, \\ a_{11}e_3 + a_{11}e_3 &= a_{33}e_3, \\ 2a_{11} &= a_{33}. \end{aligned}$$

2. When $i = 1, j = 2$;

$$\begin{aligned} d(e_1)e_2 + e_1d(e_2) &= 0, \\ (a_{11}e_1 + a_{12}e_2 + a_{13}e_3)e_2 + e_1(a_{21}e_1 + a_{22}e_2 + a_{23}e_3) &= 0, \\ a_{21}e_3 &= 0, \\ a_{21} &= 0. \end{aligned}$$

3. When $i = 1, j = 3$;

$$\begin{aligned} d(e_1)e_3 + e_1d(e_3) &= 0, \\ (a_{11}e_1 + a_{12}e_2 + a_{13}e_3)e_3 + e_1(a_{31}e_1 + a_{32}e_2 + a_{33}e_3) &= 0, \\ a_{31}e_3 &= 0, \\ a_{31} &= 0. \end{aligned}$$

4. When $i = 2, j = 1$;

$$\begin{aligned} d(e_2)e_1 + e_2d(e_1) &= 0, \\ (a_{21}e_1 + a_{22}e_2 + a_{23}e_3)e_1 + e_2(a_{11}e_1 + a_{12}e_2 + a_{13}e_3) &= 0, \\ a_{21}e_3 &= 0, \\ a_{21} &= 0. \end{aligned}$$

5. When $i = j = 2$;

$$\begin{aligned} d(e_2)e_2 + e_2d(e_2) &= 0, \\ (a_{21}e_1 + a_{22}e_2 + a_{23}e_2)e_2 + e_2(a_{21}e_1 + a_{22}e_2 + a_{23}e_3) &= 0, \\ 0 &= 0. \end{aligned}$$

6. When $i = 2, j = 3$;

$$\begin{aligned} d(e_2)e_3 + e_2 d(e_3) &= 0, \\ (a_{21}e_1 + a_{22}e_2 + a_{23}e_3)e_3 + e_2(a_{31}e_1 + a_{32}e_2 + a_{33}e_3) &= 0, \\ 0 &= 0. \end{aligned}$$

7. When $i = 3, j = 1$;

$$\begin{aligned} d(e_3)e_1 + e_3 d(e_1) &= 0, \\ (a_{31}e_1 + a_{32}e_2 + a_{33}e_1)e_1 + e_3(a_{11}e_1 + a_{12}e_2 + a_{13}e_3) &= 0, \\ a_{31}e_3 &= 0, \\ a_{31} &= 0. \end{aligned}$$

8. When $i = 3, j = 2$;

$$\begin{aligned} d(e_3)e_2 + e_3 d(e_2) &= 0, \\ (a_{31}e_1 + a_{32}e_2 + a_{33}e_2)e_2 + e_3(a_{21}e_1 + a_{22}e_2 + a_{23}e_3) &= 0, \\ 0 &= 0. \end{aligned}$$

9. When $i = 3, j = 3$;

$$\begin{aligned} d(e_3)e_3 + e_3 d(e_3) &= 0, \\ (a_{31}e_1 + a_{32}e_2 + a_{33}e_3)e_3 + e_3(a_{31}e_1 + a_{32}e_2 + a_{33}e_3) &= 0, \\ 0 &= 0. \end{aligned}$$

Then a derivation of algebra is obtained as follows :

$$d(As_{3,2}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 2a_{11} \end{bmatrix}.$$

An algebra A over K is called maximum commutative subalgebra, Com if its binary map satisfies $xy = yx$ for all $x, y \in A$. An algebra A over a field K is called maximum abelian subalgebra, n_{As} if its binary map satisfies $xy = 0$ for all $x, y \in A$. For example, let $L = span\{e_1, e_2, e_3\}$ with the following table of multiplication $A_{3,1}$: $e_1e_1 = e_3, e_1e_2 = e_3, e_2e_1 = e_3$. The algebra satisfies $e_ie_j = e_je_i$ for all $i, j = 1, 2, 3$. Then a maximum commutative subalgebra Com is $span\{e_1, e_2, e_3\}$ and also satisfies $e_ie_j = 0$ for all $i, j = 1, 2, 3$. Then we have a maximum abelian subalgebra n_{As} is $span\{e_2, e_3\}$.

In order to perform a complete list of three dimensional associative algebras, Lemma 1.1 is used to extract the coboundary, $B^2(As_2, \mathbb{C})$.

Lemma 1.1 *Let A be an n-dimensional associative algebra, and let $\{e_1, e_2, \ldots, e_m\}$ be a basis of $A^{\langle 2 \rangle}$. Then $B^2(A,K) = \langle \delta e_1^*, \delta e_2^*, \ldots, \delta e_m^*, \rangle$ where $e_i^*(e_j) = \delta_{ij}$ and δ_{ij} is the Kronecker delta.*

Proof :
Extend a basis of $A^{\langle 2 \rangle}$ to a basis $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of A. Then $\{e_1^*, e_2^*, \ldots, e_m^*\}$ form a basis of $Hom(A,K)$. Consider any $\delta f \in B^2(A,K)$, and let $f = \sum_{i=1}^{n} \alpha_i a_i^*[e_j, e_k]$ for some $\alpha_i \in K$. Then,

$$\begin{aligned}
\delta f[e_j, e_k] &= \sum_{i=1}^{n} \alpha_i e_i^*(e_j.e_k), \\
&= \sum_{i=1}^{n} \alpha_i e_i^*(\sum_{i=1}^{n} \beta_l e_l), \\
&= \sum_{i=1}^{m} \alpha_i e_i^*(e_j.e_k), \\
&= \sum_{i=1}^{m} \alpha_i \delta e_i^*(e_j.e_k).
\end{aligned}$$

Hence, $\delta f = \sum_{i=1}^{m} \alpha_i \delta e_i^*$. Further, let $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ such that $\sum_{i=1}^{m} \alpha_i \delta e_i = 0$. Then, $\sum_{i=1}^{m} \alpha_i \delta e_i[A,A] = \sum_{i=1}^{m} \alpha_i \delta e_i[A^{<2>}] = 0$. This implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$. So $\{\delta e_1, \delta e_2, \ldots, \delta e_m\}$ are linearly independent. Thus $\{\delta e_1, \delta e_2, \ldots, \delta e_m\}$ is span of $B^2(A,K)$.

This lemma is modified from (Hegazi and AbdelWaheb, 2016) in Jordan algebra case to associative algebra. It is proven that Lemma 1.1 can be apply for any algebra.

For example, let A be a two dimensional associative algebra with a basis $\{e_1, e_2\}$ and let the table of multiplication of A is given as follows :

$$A : e_1 e_1 = e_2.$$

The cocycle, $Z^2(A,K)$ is spanned by $\{\triangle_{11}, \triangle_{21}\}$.

1. $\delta e_1^* = C_{11}\triangle_{11} + C_{21}\triangle_{21}$, where $C_{11} = \delta e_1^*(e_1 e_1) = e_1^*(e_2) = 0$ and $C_{21} = \delta e_1^*(e_2 e_1) = e_1^*(0) = 0$. Then, $\delta e_2^* = span\{\}$.

2. $\delta e_2^* = C_{11}\triangle_{11} + C_{21}\triangle_{21}$, where $C_{11} = \delta e_2^*(e_1 e_1) = e_2^*(e_2) = 1$ and $C_{21} = \delta e_2^*(e_2 e_1) = e_2^*(0) = 0$. Then, $\delta e_1^* = span\{\triangle_{11}\}$.

We can conclude that the coboundary,

$$B^2(A,K) = span\{\delta e_1^*, \delta e_2^*\} = span\{\triangle_{11}\}.$$

Definition 1.16 *(Ahmed et al., 2018) An automorphism $\boldsymbol{g} : \mathbb{A} \to \mathbb{A}$ as an invertible linear map is represented by an invertible matrix*

$$g \in GL(m;\mathbb{F}) : \boldsymbol{g}(\boldsymbol{u}) = \boldsymbol{g}(eu) = egu.$$

Due to : $\quad \boldsymbol{g}(\boldsymbol{u}\cdot\boldsymbol{v}) = \boldsymbol{g}(eA(u\otimes v) = eg(A(u\otimes v)) = e(gA)(u\otimes v) \quad$ *and*

$$\boldsymbol{g}(\boldsymbol{u})\cdot\boldsymbol{g}(\boldsymbol{v}) = (egu)\cdot(egv) = eA(gu\otimes gv) = eA(gu\otimes gv) = eAg^{\otimes v}(u\otimes v),$$

the property $\boldsymbol{g}(\boldsymbol{u}\cdot\boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{u})\cdot\boldsymbol{g}(\boldsymbol{v})$ is equivalent to : $gA = Ag^{\otimes v}$. Thus, for the group of automorphisms of an algebra $\mathbb{A}$ given by MSC $A \in M(2\times 4;\mathbb{F})$ (for two dimensional algebras) one has : $Aut(A) = \{g \in GL(2;\mathbb{F}) : gA - A(g\otimes g) = 0\}$. Therefore we look only for nonsingular solutions

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of the equation $gA - A(g\otimes g) = 0$.

For example, Ahmed et al. (2018) wrote that let

$$A = A_1(\alpha_1,\alpha_2,\alpha_4,\beta_1) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{bmatrix}.$$

Due to equation $Aut(A) = \{g \in GL(2;\mathbb{F}) : gA - A(g\otimes g) = 0\}$ one has the system of equations :

$$\begin{aligned}
-ac + a\alpha_1 - a^2\alpha_1 - 2ac\alpha_2 - c^2\alpha_4 + \beta_1 &= 0,\\
-bc - b\alpha_1 - a\alpha_1 + a\alpha_2 - bc\alpha_2 - ad\alpha_2 - cd\alpha_4 &= 0,\\
a + b - ad - b\alpha_1 - ab\alpha_1 + a\alpha_2 - bc\alpha_2 - ad\alpha_2 - cd\alpha_4 &= 0,\\
-bd - b^2\alpha_1 - b\alpha_2 - 2bd\alpha_2 + a\alpha_4 - d^2\alpha_4 &= 0,\\
-ac + c\alpha_1 + 2ac\alpha_1 + c^2\alpha_2 - a^2\beta_1 + d\beta_1 &= 0,\\
-bc + bc\alpha_1 - d\alpha_1 + ad\alpha_1 + c\alpha_2 + cd\alpha_2 - ab\beta_1 &= 0,\\
c + d - ad + bc\alpha_1 - d\alpha_1 + ad\alpha_1 + c\alpha_2 + cd\alpha_2 - ab\beta_1 &= 0,\\
-bd + 2bd\alpha_1 - d\alpha_2 + d^2\alpha_2 + c\alpha_4 - b^2\beta_1 &= 0.
\end{aligned}$$

The system of equation above implies

$$a + b - ad + bc = 0,$$

i.e., $b = \triangle - a$ and

$$c + d - ad + bc = 0,$$

i.e., $d = \triangle - c$. Therefore, $\triangle = a(\triangle - c)c(\triangle - a)$ and this implies $\triangle(1 - a + c) = 0$. Since $\triangle \neq 0$ we have $c = a - 1$. The similar observation implies $d = b + 1$. Therefore we get

$$g = \begin{bmatrix} a & b \\ a-1 & b+1 \end{bmatrix}, \text{ with } \triangle = a+b \neq 0.$$

As a result the system of equation above can be rewrite a follows :

$$\begin{aligned} a^2(1+\alpha_1+2\alpha_2+\alpha_4) - a(1+\alpha_1+2\alpha_2+\alpha_4)+\alpha_4 = 0, \\ \alpha_4(a-1) = 0, \\ a^2(1-2\alpha_1-\alpha_2+\beta_4)+a(-1+\alpha_1+2\alpha_2)-\alpha_2+\alpha_1-\beta_1 = 0, \\ (\alpha_1+2\alpha_2)(a-1) = 0. \end{aligned}$$

Therefore, $Aut(\alpha_1,\alpha_2,\alpha_4,\beta_1)$=$\{I\}$.

A list of non-isomorphism algebras in two dimensional is needed before applying the Skjelbred-Sund method by using central extension to obtain the list of non-isomorphic algebras in dimension three. The method can be used to apply three dimensional algebras extend to four dimensional algebras.

Theorem 1.1 *(Graaf, 2010) In two dimensional associative algebras, there are the following non-isomorphism algebras.*
A_2^1 : *abelian;*
$A_2^2 : (e_1,e_1) = e_2;$
$A_2^3 : (e_1,e_1) = e_1, (e_1,e_2) = e_2;$
$A_2^4 : (e_1,e_1) = e_1, (e_2,e_1) = e_2;$
$A_2^5 : (e_1,e_1) = e_1, (e_1,e_2) = (e_2,e_1) = e_2;$
$A_2^6 : (e_1,e_1) = e_1, (e_2,e_2) = e_2.$

Theorem 1.2 *(Langari, 2010) In two dimensional Leibniz algebras, there are the following non-isomorphism algebras.*
L_2^1 : *abelian;*
$L_2^2 : [e_1,e_1] = e_2;$
$L_2^3 : [e_1,e_2] = -[e_2,e_1] = e_2;$
$L_2^4 : [e_1,e_2] = [e_2,e_2] = e_1.$

From (Burde, 1999) and (Rikhsiboev and Rakhimov, 2012), we have the following theorem :

Theorem 1.3 *In three dimensional Leibniz algebras, there are the following non-isomorphism algebras.*
$RRI_1 : [e_1,e_3] = -2e_1, [e_2,e_2] = e_1, [e_3,e_2] = e_2, [e_2,e_3] = -e_2;$
$RRI_2 : [e_1,e_3] = \alpha e_1, [e_3,e_2] = e_2, [e_2,e_3] = -e_2;$
$RRI_3 : [e_3,e_3] = e_1, [e_3,e_2] = e_2, [e_2,e_3] = -e_2;$
$RRI_4 : [e_2,e_2] = e_1, [e_3,e_3] = \alpha e_1, [e_2,e_3] = e_1;$
$RRI_5 : [e_2,e_2] = e_1, [e_3,e_3] = e_1;$
$RRI_6 : [e_1,e_3] = e_2, [e_2,e_3] = e_1;$
$RRI_7 : [e_1,e_3] = e_2, [e_2,e_3] = \alpha e_1 + e_2;$
$RRI_8 : [e_3,e_3] = e_1, [e_1,e_3] = e_2;$

$RRI_9 : [e_3,e_3] = e_1, [e_1,e_3] = e_1 + e_2;$
$L_3 : [e_1,e_2] = e_3, [e_1,e_3] = 2e_1, [e_2,e_1] = -e_3, [e_2,e_3] = 2e_2,$
$\quad [e_3,e_1] = -2e_1, [e_3,e_2] = 2e_2;$
$L_8(\alpha) : [e_1,e_2] = e_3, [e_1,e_3] = \alpha e_3, [e_2,e_1] = -e_2, [e_3,e_1] = \alpha e_3;$
$L_9 : [e_1,e_2] = e_2, [e_2,e_1] = -e_2;$
$L_{10} : [e_1,e_2] = e_2, [e_1,e_3] = e_2 + e_3, [e_2,e_1] = -e_2, [e_3,e_1] = -e_2 - e_3;$
$L_{15} : [e_1,e_2] = e_2, [e_1,e_3] = e_3, [e_2,e_1] = -e_2, [e_3,e_1] = -e_3;$
$L_{16} : [e_1,e_2] = e_3, [e_2,e_1] = -e_3;$

1.2 Problem Statement

The classification problem of algebras is a rather complicated problem especially for higher dimensions. As well known, the list of isomorphism classes for low dimensional algebras such as associative, Lie and Leibniz algebras have been known. Does the classification of algebras in higher dimensional can be arising from the list of algebras in low dimensional? One of the methods is the Skjelbred-Sund method that has been used in the case of Lie algebra. The study in this thesis is to investigate this method can be applied to obtain the list of isomorphism classes for higher dimensional associative and Leibniz algebras cases.

1.3 Research Objectives

This research objectives are stated as follows :

1. To find the cocycle of associative and Leibniz algebras in dimension two and three.
2. To establish the group of automorphisms of associative and Leibniz algebras in dimension two and three.
3. To classify nilpotent associative algebras using central extension in three dimensional.
4. To classify nilpotent Leibniz algebras using central extension in three and four dimensional.

1.4 Methodology

Throughout this research, a classification procedure is developed to lead directly to our objectives.

The classification procedures are as follows:

- For a given algebra of smaller dimension, firstly list its center in order to identify the 2-cocycle that satisfies

$$\theta^{\perp} \cap C(A) = 0.$$

- Determine the 2-cocycle, the 2-coboundary and compute the quotient $H^2(A,K)$. For each algebra A with given basis $\{e_1, e_2, ..., e_s\}$, a 2-cocycle θ is presented by a matrix (e_{ij}), $\theta = \sum_{i,j=1}^{n} c_{ij} \triangle_{ij}$, where $\triangle_{ij}$ is the $n \times n$ matrix with (ij) element being 1 and all the others 0. To compute the 2-cocycle, all the constraints on the elements c_{ij} of the matrix (c_{ij}) are listed.

- Find a (maybe a redundant) list of representatives of the orbits of $Aut(A)$ acting on the θ.

- For each θ found, construct A_{θ}. Discard the isomomorphic ones.

1.5 Outline of Contents

This section gives a brief outline of the material and main concepts in each chapter. There are five chapters in this thesis with the following contents:

Some basic concepts of algebra in Chapter 1 is described. It started with the definition of associative algebra and Leibniz algebra together with their examples. Then followed by other definitions, examples, lemmas and theorems. Our main objectives and a methodology of this research are listed.

Chapter 2 presents a literature review of the research. This part gives a deep understanding regarding the backgrounds of the research which is introduced by Louis Loday in 1977.

The classification of associative algebras in three dimensional is stated in Chapter 3. Then several extension invariants is presented to distinguish isomorphism classes. The classification is shown in details by including a group action. Then, a list of algebras are identified.

In Chapter 4, three dimensional and four dimensional Leibniz algebras are classified. The similar method as mentioned in Chapter 3 is presented. As requested, the list of isomorphism classes in dimensions three and four of Leibniz algebras are presented. Then several extension invariants are considered to distinguish isomorphism classes. The procedure of the classification is shown including group action in details. At the end, a list of algebras is identified.

Finally, Chapter 5 provides the summary of all contributions that has been done in this thesis and also discuss on future research.

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