UNIVERSITI PUTRA MALAYSIA

MULTISTEP BLOCK METHOD FOR SOLVING SECOND AND THIRD ORDER BOUNDARY VALUE PROBLEMS WITH ROBIN AND MIXED TYPE BOUNDARY CONDITIONS

NADIRAH BINTI MOHD NASIR

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By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To my beloved parents,
who gave their children the opportunities to develop a passion of learning,
and
to my families,
for all the Du'a along this journey,
with lots of love, thank you so much.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chairman : Zanariah Abdul Majid, PhD
Faculty : Institute for Mathematical Research

This thesis investigates on the numerical solutions for solving two-point and multipoint boundary value problems (BVPs) subject to Robin and mixed boundary conditions. The BVPs are solved directly using the new developed two-point diagonally implicit multistep block method in the form of Adams type formula.

Constant and variable step size strategy are employed for solving two-point second-order BVPs. Meanwhile, the computed solutions for two-point and multipoint third-order BVPs are limit to constant step size. Shooting technique is implemented in order to solve the BVPs. The initial estimate values are obtained using the Newton's divided difference interpolation method and Steffensen's method. Alternatively, the first derivative function is absence during the calculation of guessing values compared to the shooting technique via the Newton's method.

The analysis included order, error constants, consistency, zero-stability and convergence are presented in describing the characteristics of the proposed methods. All the computational procedures were undertaken using the C language in a Code::Blocks 16.01 cross platform.

Numerical results showed significant findings where the proposed methods could offer better accuracy results, less costly in terms of total function calls and faster in timing compared to the existing methods.

In conclusion, the proposed methods and developed algorithms were shown to be a reliable BVPs solver for solving two-point and multipoint BVPs subject to Robin and mixed boundary conditions directly.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH BLOK MULTILANGKAH UNTUK PENYELESAIAN MASALAH NILAI SEMPADAN PERINGKAT KEDUA DAN KETIGA DENGAN SYARAT SEMPADAN JENIS ROBIN DAN BERCAMPUR

Oleh

NADIRAH BINTI MOHD NASIR

Julai 2020

Pengerusi : Zanariah Abdul Majid, PhD
Fakulti : Institut Penyelidikan Matematik

Tesis ini mengkaji tentang penyelesaian berangka untuk menyelesaikan masalah nilai sempadan (MNS) dua titik dan berbilang titik yang memenuhi syarat sempadan bersifat Robin dan bercampur. MNS diselesaikan secara terus dengan menggunakan kaedah yang baru dibentuk iaitu kaedah blok multilangkah dua-titik tersirat pepenjuru berdasarkan formula Adam.

Strategi saiz langkah tetap dan berubah digunakan untuk menyelesaikan MNS dua titik peringkat kedua. Sementara itu, penyelesaian MNS dua titik dan berbilang titik peringkat ketiga dikomputasi dengan menghadkan kepada saiz langkah tetap sahaja. Teknik tembakan dilaksanakan untuk memperoleh penyelesaian MNS. Nilai-nilai anggaran awal diperolehi dengan menggunakan kaedah interpolasi perbezaan terbahagi Newton dan kaedah Steffensen. Secara alternatif, fungsi terbitan pertama tidak wujud dalam penggiraan nilai-nilai tekaan berbanding penggunaan teknik tembakan melalui kaedah Newton.

Analisis termasuk ciri-ciri peringkat, pemalar ralat, konsistensi, kestabilan-sifar dan penumpuan dibincangkan bagi menghuraikan ciri-ciri kesemua kaedah yang dicadangkan. Keseluruhan prosedur pengiraan secara komputasi telah dijalankan dengan menggunakan bahasa pengaturcaraan C dalam perisian Code::Blocks 16.01.

Keputusan berangka menunjukkan penemuan penting bahawa kaedah-kaedah yang

dicadangkan dapat mencapai kejituan keputusan yang lebih baik, lebih jimat dari segi jumlah fungsi panggilan dan memperoleh masa yang pantas berbanding kaedah sedia ada.

Kesimpulannya, kaedah yang dicadangkan dan algoritma yang dibangunkan berupaya menjadi penyelesai MNS yang boleh dipercayai untuk menyelesaikan MNS dua titik dan berbilang titik yang memenuhi syarat sempadan bersifat Robin dan bercampur secara kaedah terus.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

Zanariah Abdul Majid, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Fudziah Ismail, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Norfifah Bachok @ Lati, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

ZALILAH MOHD SHARIFF, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 08 October 2020

Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________________
Name of Chairman of Supervisory Committee:
Prof. Dr. Zanariah Abdul Majid

Signature: ______________________________
Name of Member of Supervisory Committee:
Prof. Dr. Fudziah Ismail

Signature: ______________________________
Name of Member of Supervisory Committee
Assoc. Prof. Dr. Norfifah Bachok @ Lati

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

h	step size
BCs	Boundary Conditions
BVPs	Boundary Value Problems
IVPs	Initial Value Problems
ODEs	Ordinary Differential Equations
bvp4c	Matlab solver with fourth order collocation method
bvp5c	Matlab solver with fifth order collocation method
2PDD4	Direct two point diagonally block method of order four
2PDD5	Direct two point diagonally block method of order five
2PDD6	Direct two point diagonally block method of order six
2DDM4	Direct two point diagonally block method of order four for solving third order BVPs
2DDM5	Direct two point diagonally block method of order five for solving third order BVPs
2DDM6	Direct two point diagonally block method of order six for solving third order BVPs
2PDVS4	Direct two point diagonally block method of order four using variable step size strategy
2PDVS5	Direct two point diagonally block method of order five using variable step size strategy

CHAPTER 1

INTRODUCTION

1.1 Background

The behaviour of many physical problems appear frequently in terms of differential equations. Those behaviour can be either position-dependent changes or time-dependent changes. Differential equations are extremely important in vast field of studies including mathematics, physics, chemistry, economics as well as engineering. The solution to the differential equations can be obtained using analytical and numerical approaches. Analytical solution will result in exact value and it is almost accurate. However, in dealing with mathematical model in the form of nonlinear and complex differential equations, analytical solution sometimes is impractical and has a limitation to give the required solution. Numerical solution consists of approximate value together with the numerical error. This errors are important to be taken into consideration while computing the numerical results because it will tell us how accurate the numerical answers within a specified tolerance. In addition, if the exact solution is provided to the differential equations problems, then it is useful to validate the reasonable numerical results produce by the numerical methods.

Ordinary differential equations (ODEs) and partial differential equations (PDEs) are two major types of differential equations. Both equations represent the relationship between the unknown function with its derivative. This thesis will tackle the differential equations problems in the form of ODEs rather than PDEs.

Generally, a D^{th} order ordinary differential equation is given by

$$\frac{d^D y}{dx^D} = f\left(x, y, y', y'', y^{(3)}, \dots, y^{(D-1)}\right),\ x \in [x_0, x_N]. \tag{1.1.1}$$

To obtain the desired solution of (1.1.1) to the function, $y(x)$, D conditions are required. These conditions can be specified as either initial conditions or boundary conditions. Numerically, by solving higher order of (1.1.1) directly without converting to the first order system will minimizing the computational cost of process and save in time-consuming.

In numerical method, solving (1.1.1) involves computation of consecutive approximations, y_{n+i}, and the function, f_{n+i} for $i = 0, 1, \dots, k$. This procedure constitutes one-step method and multistep method, corresponds to $k = 1$ and $k > 1$, respectively. One-step method permit calculation of the approximation using only one previous value, while multistep method requires more than one previous values. This thesis focusing on multistep method with one-step method will be employed to give the appropriate number of starting values in order to initiate the multistep procedure since the implementation is not a self-starting method.

1.2 Boundary Value Problems

In a nutshell, a boundary value problem is a differential equation, typically in one dimension is an ODE which has values assigned on the physical boundary of the domain and deals with higher order differential equations. Consider a general form of second-order BVPs for $D = 2$ in (1.1.1) as

$$y''(x) = f(x, y, y'),\ x \in [a, b] \tag{1.2.1}$$

subject to two-point boundary conditions (BCs)

$$c_1 y(a) + c_2 y'(a) = \alpha \quad \text{and} \quad c_3 y(b) + c_4 y'(b) = \beta \tag{1.2.2}$$

where a, b, α, β and c_i for $i = 1, 2, 3, 4$ are all constants.

There are three types of boundary conditions appear oftenly related to BVPs. If only functional value of the solution specified in (1.2.2), then this condition represents **Dirichlet BCs** and can be written as

$$c_1 y(a) = \alpha \quad \text{and} \quad c_3 y(b) = \beta \tag{1.2.3}$$

corresponds to the case of $c_2 = c_4 = 0$. Besides, **Neumann BCs** correspond to the case of $c_1 = c_3 = 0$ as

$$c_2 y'(a) = \alpha \quad \text{and} \quad c_4 y'(b) = \beta \tag{1.2.4}$$

where only derivative of the solution is given in (1.2.2). Finally, if both information exist and specified in the form of linear combination between functional value and derivative of the solution, hence (1.2.2) is known as **Robin BCs**

$$c_1 y(a) + c_2 y'(a) = \alpha \quad \text{and} \quad c_3 y(b) + c_4 y'(b) = \beta. \tag{1.2.5}$$

This thesis concentrates on solving (1.2.1) that associated with Robin type conditions occur at the boundary point and subsumes under the following types:

- Type 1: The Robin boundary conditions as given in (1.2.5).

- Type 2: The mixed set of Robin and Dirichlet boundary conditions given as

$$c_1 y(a) + c_2 y'(a) = \alpha \quad \text{and} \quad c_3 y(b) = \beta. \tag{1.2.6}$$

- Type 3: The mixed set of Robin and Neumann boundary conditions given as

$$c_1 y(a) + c_2 y'(a) = \alpha \quad \text{and} \quad c_4 y'(b) = \beta. \tag{1.2.7}$$

- Type 4: The mixed set of Dirichlet and Robin boundary conditions given as

$$c_1 y(a) = \alpha \quad \text{and} \quad c_3 y(b) + c_4 y'(b) = \beta. \tag{1.2.8}$$

The detail discussion on the third-order BVPs will be highlighted in Chapter 4 and 5 since the differential equation problems will involve a multipoint boundary conditions.

There are several references such as Usmani (1972), Chawla (1978) and Bialecki (1991) defined conditions in (1.2.5) as mixed boundary conditions. However, our preference is Robin BCs instead of mixed BCs. This is supported by the definition stated in Gustafson and Abe (1998), "mixed boundary condition normally means that on one portion of the boundary, you have one of the three usual BCs, whereas on another part of the boundary you have a different one". Lawley and Keener (2015) mentioned that the Robin boundary condition specifies a relationship between the solution and its derivative.

The real function, f in (1.2.1) is assumed to satisfy the Lipschitz condition as follows

$$\begin{aligned} |f(x,v_1,w) - f(x,v_2,w)| &\le K|v_1 - v_2|, \\ |f(x,v,w_1) - f(x,v,w_2)| &\le K|w_1 - w_2| \end{aligned} \tag{1.2.9}$$

for all points $(x,v_i,w),(x,v,w_i), i = 1,2$ in the set

$$D = \{f(x,v,w)|a \le x \le b, -\infty < v,w < \infty\}.$$

Theorem 1.1 *(Atkinson et al., 2011)*
For the given problem in (1.2.1) *with* (1.2.5)*, assume $f(x,v,w)$ to be continuos on the set, D and satisfies the Lipschitz condition in* (1.2.9)*. In addition, on the set D, f satisfies the followings:*

1. $f_v(x,v,w) > 0$,
2. $\left|f_w(x,v,w)\right| \le K$ *for some constant,* $K > 0$,
3. *For the boundary conditions of* (1.2.5)*, assume*

$$\begin{aligned} c_1c_2 \geqslant 0, \quad c_3c_4 \geqslant 0, \\ |c_1| + |c_2| \neq 0, \quad |c_3| + |c_4| \neq 0, \quad |c_1| + |c_3| \neq 0. \end{aligned}$$

Then, the BVPs given by (1.2.1) *associated with* (1.2.5) *has a unique solution.*

In this study, linear and nonlinear BVPs will be solved using systematic iterative approach via the shooting method. Since there is a missing initial condition involved in the given BCs, then solving BVPs using shooting technique required initial guessing to start the procedure.

1.3 Objectives of the Thesis

This thesis will be focused on the following objectives:

1. to derive two-point diagonally implicit multistep block method formula for solving non-stiff two-point and multipoint boundary value problems subject to Robin and mixed type boundary conditions;
2. to analyze the properties of the method including order, stability, consistency and convergence in details;
3. to establish the shooting strategy via Newton's divided difference interpolation method and Steffensen's method for solving second-order and third-order BVPs using C programming source code;
4. to develop the algorithms for the multistep block method with constant step size and variable step size strategy adapted with the shooting technique.

1.4 Motivation

The development on the direct multistep block method is well established for enabling to solve various types of differential equations. However, the implementation and performances on numerical algorithm for use in boundary value problems with Robin boundary conditions has not been extensively studied.

Numerous works have been carried out to investigate the BVPs subject to Dirichlet and Neumann boundary conditions using block method adapted with the shooting technique for solving directly the aforementioned BVPs. Moreover, majority of the previous work used Newton's method as the strategy to improvise the guessing values during the shooting procedure.

Therefore, motivation of this study is to develop the two-point diagonally multistep block method with different order between first point and second point. The main aim of the numerical integration scheme development using the diagonally formula is to demonstrate in practice that the diagonally multistep block method is significantly cheaper in computational effort and favourably competitive with existing methods. In addition to that, the diagonally formula is sensible in order to preserve the high accuracy of the computed results. Meanwhile, this thesis will emphasis on the shooting strategy adapted with the Newton's divided difference interpolation method and Steffensen's method as the iterative scheme for improvise the guessing values. This iterative scheme is different from the Newton's method because the former approach used interpolation while the latter approach is a derivative free method.

1.5 Scope of the study

This thesis concentrates on the numerical solution of two-point diagonally multistep block method for solving second-order and third-order boundary value problems associated with Robin and mixed boundary conditions. In solving second order BVPs, the condition imposed on the independent variable at two different values. In contrast, solving third order BVPs deals with the boundary conditions define at two-point boundary conditions and multipoint boundary conditions.

Two-point diagonally multistep block method of order four, five and six will be implemented to solve all the aformentioned Robin and mixed boundary value problems via the shooting technique. The shooting technique adapted with the Newton's divided difference interpolation method and Steffensen's method are employed for generating the guessing values. In addition, the numerical results for second order BVPs will be generated using fixed and variable step size. Meanwhile, the numerical results for third order BVPs will limit to constant step size only.

The analysis of the methods including order, consistency, zero-stability, convergence and stability are also discussed in this thesis.

1.6 Outline of the Thesis

The organization of the thesis are as follows.
Chapter 1 provides a brief introduction to the boundary value problems, describing the type of boundary conditions and explanation on the prelimenary concept and theory that devoted to BVPs.

Chapter 2 highlights on the important definitions that necessary in describing the properties of the developed methods. This chapter also cover on the theory of Lagrange interpolation polynomial that will be used during the derivation of the direct integration formula. The chronological studies that leads to a road map of this research will be part of this chapter as a briefly reviewed from the previous works.

Chapter 3 emphasizes on the derivation of two-point diagonally multistep block method of order four, five and six with constant step size. These block methods will be used to solve second order two-point boundary value problems subject to Robin and mixed boundary conditions. The shooting technique adapted with Newton's divided difference interpolation method and Steffensen's method are also introduced in this chapter. All the analysis properties related to the derived methods also included in this chapter.

Chapter 4 focuses on the direct integration formula to solve the third order boundary value problems associated with Robin boundary conditions. Again, the derivation is in the form of two-point diagonally multistep block method of order four, five and six with constant step size. All the analysis properties related to the derived formula also included as part of this chapter.

Chapter 5 concerns on the implementation to tackle the third order multipoint boundary value problems using the direct block methods developed in Chapter 4. At the begining, this chapter higlights on the two distinct algorithms to solve respective BVPs since there are three classifications on the types of multipoint boundary conditions that have been considered in this thesis.

Chapter 6 discusses on the derivation of two-point diagonally multistep block method using a variable step size strategy. This chapter present on the strategy to choose the step size adjustment. This adjustment correspond to three different decision which are keeping the new step size remain constant, or double or halved, from the previous step size.

Finally, Chapter 7 concludes the important findings from this research study. At the same time, some potential recommendations for further research works will be highligted in this chapter.

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