UNIVERSITI PUTRA MALAYSIA

OPTION PRICING FOR ROUGH HESTON MODEL USING NUMERICAL METHODS

SIOW WOON JENG

IPM 2022 1

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By

SIOW WOON JENG

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

November 2021

DEDICATIONS

Special dedicated to:

My father & my Mother

Siblings & Friends

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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November 2021

Chairman: Professor Adem Kiliçman, PhD
Institute: Mathematical Research

The value of an option is largely affected by the underlying assumptions or models, such as the modelling of the volatility process. Fractional Brownian motion has been shown to be able to accurately model and forecast volatility processes displayed in the financial market. The key attribute of modelling the empirical volatility using the fractional Brownian motion is its rough movement nature which is governed by a parameter called Hurst parameter H with the valid range of $H \in (0, 0.5)$ to display the roughness effect. In response to the development, we study the option pricing methods of rough volatility model to price derivatives such as the widely acceptable option–S&P 500 (SPX) option.

This thesis will focus on the option pricing methods under a particular rough volatility model called rough Heston model. The main problem of this study is that the characteristic function of the rough Heston model contains a fractional Riccati equation which has no closed-form solution. Solving the fractional Riccati equation using the standard iterative method (fractional Adams-Bashforth-Moulton method) would require $O(N^2)$ time complexity where N is the number of steps of the standard method. If N_c is the number of steps in the numerical integration of the Fourier inversion method, the computational cost would further increase to $O(N^2 N_c)$ time complexity when fractional Adams-Bashforth-Moulton is used as the medium to price option under rough Heston model. The huge computational cost on the computation of option price under the rough Heston model would undoubtedly be a barrier to most practitioners. The main objectives of this study are to improve an existing approximation method called Padé approximant to approximate fractional Riccati equation's solution and construct an approximation formula for option price without involving the characteristic function of rough Heston model.

The main contribution of our study is that we have modified and improved an existing Padé approximant such that it can accurately approximate the solutions of fractional Riccati equation on the Hurst parameter range of $H \in (0, 0.5)$ unlike the Padé approximant from previous study where its accuracy will increasingly deteriorate when the Hurst parameter H increases up to 0.5. The time complexity of modified Padé approximant is kept at $O(1)$ time complexity. In addition, we have also constructed an approximation option pricing formula under rough Heston model. Specifically, the method utilises the decomposition formula of option price under certain stochastic volatility, and depending on the structure of forward variance curve used, the approximation formula would require $O(1)$ or $O(n_f)$ time complexity to compute the option value where n_f is the number of integration steps. The result of the numerical experiment has shown that the methods are capable of matching the SPX options very accurately in a non-extreme market condition and moderately accurate in an extreme market condition, and most importantly, the option pricing method can be computed in a time-efficient manner.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENILAIAN OPSYEN BAGI MODEL HESTON KASAR MENGGUNAKAN KAEDAH BERANGKA

Oleh

SIOW WOON JENG

November 2021

Pengerusi: Profesor Adem Kiliçman, PhD
Institut: Penyelidikan Matematik

Nilai sebuah opsyen besar dipengaruhi oleh andaian atau model yang mendasari, seperti pemodelan proses turun naik. Gerakan pecahan Brownian telah terbukti dapat memodelkan dan meramalkan proses turun naik yang dipaparkan di pasaran kewangan dengan tepat. Atribut utama memodel turun naik empirikal menggunakan pecahan gerakan Brownian adalah sifat pergerakan kasarnya yang dikawal oleh parameter yang dikenali sebagai parameter Hurst H dengan julat sah $H \in (0, 0.5)$ untuk memaparkan kesan kekasaran. Berdasarkan tindak balas perkembangannya, pengkaji mengkaji kaedah penentuan opsyen harga terhadap model turun naik kasar kepada harga derivatif seperti opsyen yang boleh diterima secara meluas–opsyen S&P 500 (SPX).

Tesis ini memberi tumpuan kepada kaedah penentuan harga opsyen di bawah model turun naik kasar tertentu yang dikenali sebagai model Heston kasar. Masalah utama kajian ini adalah mengenai fungsi ciri model Heston kasar yang mengandungi persamaan pecahan Riccati yang tidak mempunyai penyelesaian bentuk tertutup. Menyelesaikan persamaan pecahan Riccati menggunakan kaedah lelaran piawai (kaedah pecahan Adams-Bashforth-Moulton) memerlukan kerumitan masa $O(N^2)$ dengan N ialah bilangan langkah kaedah piawai. Jika N_c ialah bilangan langkah dalam penyepaduan berangka kaedah penyongsangan Fourier, kos pengiraan akan terus meningkat kepada kerumitan masa $O(N^2 N_c)$ apabila pecahan Adams-Bashforth-Moulton digunakan sebagai pilihan sederhana kepada harga di bawah model kasar Heston. Kos pengiraan yang besar pada pengiraan harga opsyen di bawah model kasar Heston sudah pasti akan menjadi penghalang kepada kebanyakan pengamal. Objektif utama kajian ini adalah untuk menambah baik kaedah anggaran sedia ada yang dikenali sebagai anggaran Padé untuk menyelesaikan anggaran per-

samaan pecahan Riccati dan membina rumus penghampiran untuk harga opsyen tanpa melibatkan fungsi ciri model Heston kasar.

Sumbangan utama kajian ini ialah pengkaji telah mengubah suai dan menambah baik anggaran Padé sedia ada supaya ia boleh menyelesaikan anggaran persamaan pecahan Riccati dengan tepat pada julat parameter Hurst $H \in (0, 0.5)$ tidak seperti anggaran Padé daripada kajian lepas yang mana ketepatannya akan semakin merosot apabila parameter Hurst H meningkat sehingga 0.5. Kerumitan masa anggaran Padé yang diubah suai dikekalkan pada kerumitan masa $O(1)$. Di samping itu, pengkaji telah membina rumus penentuan harga pilihan anggaran di bawah model kasar Heston. Secara khusus, kaedah ini menggunakan rumus penguraian harga opsyen di bawah turun naik stokastik tertentu dan bergantung pada penggunaan struktur lengkung varians hadapan iaitu rumus penghampiran memerlukan kerumitan masa $O(1)$ atau $O(n_f)$ untuk mengira nilai opsyen yang mana n_f adalah bilangan langkah integrasi. Keputusan eksperimen berangka telah menunjukkan bahawa kaedah tersebut mampu memadankan pilihan SPX dengan sangat tepat dalam keadaan pasaran yang tidak ekstrim dan sederhana tepat dalam keadaan pasaran melampau dan yang paling penting, pilihan kaedah penentuan harga yang boleh dikira dengan cara masa efisien.

ACKNOWLEDGEMENTS

Firstly, I would like to express my most sincere gratitude to my supervisor, Prof. Dr. Adem Kiliçman, for his endless support, supervision, encouragement, and assistance in the journey of conducting research. His insightful feedbacks and guidance have helped me to identify the key components of my research work and enabled me to improve my research work.

Furthermore, I want to acknowledge Associate Prof. Dr. Siti Nur Iqmal Ibrahim as committee member for her assistance, effort, time, and valuable suggestions for the completion of this thesis.

I am also deeply grateful for the financial support provided by Ministry of Higher Education, Malaysia, with the project number of FRGS/1/2018/STG06/UPM/01/3 from the Fundamental Research Grant Scheme (FRGS).

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

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LIST OF ABBREVIATIONS

fBm	Fractional Brownian Motion
PDE	Partial Differential Equation
ATM	At-The-Money
rBergomi	Rough Bergomi model
BSS	Brownian Semi-stationary
SABR	Stochastic Alpha-Rho-Beta
PECE	Predictor-Evaluate Correct Evaluate
MPA	Multipoint Padé approximation method or approximant
GPadé	Third-Order Padé Approximant proposed by Gatheral
FPadé	Fourth-Order Padé Approximant
LADM	Laplace-Adomian decomposition method
FRE	Fractional Riccati equation
FAM	Fractional Adams-Bashforth-Moulton method
OGPadé	Rough Heston's Option Price where its fractional Riccati equation is approximated by GPadé
OFPadé	Rough Heston's Option Price where its fractional Riccati equation is approximated by FPadé

CHAPTER 1

INTRODUCTION

1.1 Basics Concept and History

A stock is a security that can be bought and sold by the general public or private investors. It represents the ownership of a fraction of the corporation that issued the stock/share. Stocks are issued mainly to raise capital from public or private investors to fund their business activities. In return, the owners of the stocks are entitled to receive the particular corporation's assets or profits based on the proportion of the company value/stock they own. The value of the stock strongly depends on the current performance and future aspects of the company. Under normal circumstances, investors can decide the time of the liquidation of the stock to make profit or loss. With that being said, the Dutch East India Company becomes the first company to be publicly traded in the year 1602 (Beattie, 2021). Noticeably, stock trading has been increasingly popular with the increasing presence of internet.

Call and put options are financial derivatives that offer holder the right, but not the obligation to buy or sell an asset at a specified time and strike price. The name "financial derivative" comes from its function as its value is determined based on its underlying asset, e.g. stock and volatility of the stock. Interestingly, the use of options contracts dates back to ancient times where the transaction contracts with embedded option features would be used to commerce goods (Poitras, 2009). Eventually, the free-standing option contracts were developed to accustomed in exchange trading from the period 16th to 18th century. In particular, options traded in London's Exchange Alley were an essential activity during the late 17th century. The major change to options trading happened in the late 20th century where the stock exchanges transitioned its physical trading activities to electronic system. This event subsequently created the wide accessibility of options trading for both local and international investors.

Depending on how the option is used, it can be both a risk management tool or a speculative device to the investors. One particular feature as a risk management tool is its insurance feature in which would limit the major uprise or downside movement of the underlying assets' value. At the same time, it is noted that the option costs lesser than the stock price itself and it has the potential of granting the holder its underlying asset in the future, therefore, some speculators would prefer to purchase options when they anticipate the stock will move by a large magnitude of the predetermined strike price.

Modelling stock price accurately to price option has been an ongoing problem for several decades, it is mainly due to the fact that the stochastic patterns of the volatility in the financial market are difficult to model. Black-Scholes model (Black & Scholes, 1973) is the most well-known model that prices the market option, but it has several unrealistic assumptions. In particular, the fluctuation of the value of stock or share does not follow the log-normal distribution. The Black Monday crash in the year 1987 reflects Black-Scholes model's unrealistic assumptions, i.e. the change of stock price has a fat tail (large unexpected movement happened more frequently than what previously expected). Furthermore, the model's implied volatility that should by right reflects the stock market's volatility does not remain as a constant value as contrary to the Black-Scholes model's constant volatility assumption.

This thesis specifically focuses on a stochastic volatility model called rough Heston model and its option pricing method. We will first go through several definitions that are related to options. The following definitions can be found in a book written by Hull (2003).

Definition 1.1 *A European call option on an asset S_t paying no dividends, with maturity date T and strike price K is defined as a contingent claim with payoff $(S_T - K)^+$.*

Definition 1.2 *A European put option on an asset S_t paying no dividends, with maturity date T and strike price K is defined as a contingent claim with payoff $(K - S_T)^+$.*

Definition 1.3 *The stock price S_t is the price specified (at that particular time t) in the call (put) contract at which it is related as $(S_T - K)^+$ and $(K - S_T)^+$ for call and put option respectively.*

Definition 1.4 *The exercise or strike price K is the price specified in the call (put) contract at which the asset may be bought (sold) when exercised.*

Definition 1.5 *The expiration or maturity date T is the date specified in the call (put) contract at which the contract reaches the end of the life of the contract, and the holder of the option will obtain the right but not the obligation to exercise the option.*

Definition 1.6 *The volatility σ of the stock is a measure of uncertainty about the returns provided by the stock.*

Definition 1.7 *The risk-free interest rate r is the rate of an asset that is perceived as no risk of default. Derivatives traders would normally use interest rates implied by* *bonds and Treasury bills, or even perhaps LIBOR or overnight indexed swap rates as the risk-free interest rate. LIBOR is an acronym for London Interbank Offered Rate and it is determined once a day by the British Bankers' Association as a reference interest rate.*

Definition 1.8 *Intrinsic value of an option is defined as the maximum of zero and the value of the option if it were exercised immediately, i.e.* $\max(S_t - K, 0)$ *for call option and* $\max(K - S_t, 0)$ *for put option where* S_t *is the current asset price at time* t *and* K *is the strike price.*

Definition 1.9 *In-the-money call option is when the current asset price is greater than the strike price. Oppositely, the in-the-money put option is when the current asset price is less than the strike price. The in-the-money call (put) option can also be defined as the positive (negative) intrinsic value of the option.*

Definition 1.10 *At-the-money option is when the current asset price and strike price are equivalent. At-the-money option can also be defined as zero intrinsic value of the option.*

Definition 1.11 *Out-of-the-money call option is when the current asset price is less than the strike price. Oppositely, the out-of-the-money put option is when the current asset price is greater than the strike price. The out-of-the-money call (put) option can also be defined as the negative (positive) intrinsic value of the option.*

Definition 1.12 *Short selling or commonly referred to as "shorting" is the action of selling an asset that is not owned by the individual or party. The individual or party has to eventually close the "short" position by buying back the asset to return it to the borrowed party. A fee may be charged for lending the securities or shares to the shorting party.*

1.2 Stochastic Processes and Brownian Motion

Stochastic processes are the mathematical models that would appear and behave in a random manner. They are commonly used in many fields, such as physics, chemistry, computer science and biology. The "unpredictable" changes in stock and volatility have greatly attracted the use of stochastic processes in finance. In this subsection, we introduce the basics of stochastic processes and the most commonly used stochastic process, i.e. Brownian motion. The material below will serve as the prerequisites to build the stochastic calculus in the following section. The following definitions and theorem can be found in a book written by Shreve (2004) unless specified otherwise.

Definition 1.13 *Let Ω be a nonempty set and let $\mathscr{F}$ be a collection of subsets of Ω. We refer to $\mathscr{F}$ as a σ-algebra (or σ-field) provided that:*

1. *the empty set $\emptyset$ belongs to $\mathscr{F}$,*
2. *whenever a set A belongs to $\mathscr{F}$, its complement A^c also belongs to $\mathscr{F}$, and*
3. *whenever a sequence of sets $A_1, A_2, \ldots$ belongs to $\mathscr{F}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathscr{F}$.*

Definition 1.14 *Let $\mathbb{P}$ be a probability measure such that it functions to assign a number in $[0,1]$ to every set $A \in \mathscr{F}$, and it is called the probability of A and written as $\mathbb{P}(A)$. We require:*

1. $\mathbb{P}(\Omega) = 1$, *and*
2. *(countable additivity) whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathscr{F}$, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Remark 1.1 *The triple $(\Omega, \mathscr{F}, \mathbb{P})$ is known as a probability space.*

Definition 1.15 *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A random variable is a real-valued function X defined on Ω with the property that for every Borel subset B of $\mathbb{B}$, the subset of Ω given by*

$$\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$$

is in the σ-algebra $\mathscr{F}$. It is permissible for a random variable to take values $+\infty$ and $-\infty$.

Definition 1.16 *Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0,T]$ there is a σ-algebra $\mathscr{F}(t)$. Assume further that if $s \leq t$, then every set in $\mathscr{F}(s)$ is also in $\mathscr{F}(t)$. Then we call the collection of σ-algebras $\mathscr{F}(t)$, $0 \leq t \leq T$, a filtration. We will frequently simplify the notation to $\mathscr{F}_t := \mathscr{F}(t)$.*

Definition 1.17 *Let X be a random variable defined on a nonempty sample space Ω. Let $\mathscr{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ is also in $\mathscr{G}$, we say that X is $\mathscr{G}$-measurable.*

Definition 1.18 *Let Ω be a nonempty sample space equipped with a filtration $\mathscr{F}(t)$, $0 \leq t \leq T$. Let $X(t)$ be a collection of random variables indexed by $t \in [0,T]$. We say this collection of random variables is an adapted stochastic process if, for each t, the random variable $X(t)$ is $\mathscr{F}(t)$-measurable. We denote $X_t := X(t)$ for simplicity purposes.*

Definition 1.19 *Let X be a random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The expectation of X is defined as*

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$$

only if X is integrable, i.e.

$$\mathbb{E}[|X|] = \int_{\Omega} |X(\omega)| d\mathbb{P}(\omega) < \infty.$$

Definition 1.20 *Let X be a random variable whose expected value is defined. The variance of X denoted as $Var(X)$ is as follows*

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Note that, the standard deviation of X is $\sqrt{Var(X)}$.

Definition 1.21 *Let X and Y be random variables whose expectations are defined. The covariance of X and Y is*

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Furthermore, the correlation coefficient of X and Y, $\rho(X,Y)$ has the relationship of

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}},$$

where $Var(X)$ and $Var(Y)$ are the variances of X and Y.

Definition 1.22 *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathscr{F}_t$, $0 \leq t \leq T$, be a filtration of a sub-σ-algebras of $\mathscr{F}$. Consider an adapted stochastic process M_t, $0 \leq t \leq T$.*

1. *If*

$$\mathbb{E}[M_t|\mathscr{F}_s] = M_s \quad \text{for all } 0 \leq s \leq t \leq T,$$

then we say that this process is a martingale, i.e. it has no tendency to rise or fall.

2. *If*

$$\mathbb{E}[M_t|\mathscr{F}_s] \geq M_s \quad \text{for all } 0 \leq s \leq t \leq T,$$

then we say that this process is a submartingale, i.e. it has no tendency to fall, but it may have a tendency to rise.

3. *If*

$$\mathbb{E}[M_t|\mathscr{F}_s] \leq M_s \quad \text{for all } 0 \leq s \leq t \leq T,$$

then we say that this process is a supermartingale, i.e. it has no tendency to rise, but it may have a tendency to fall.

Definition 1.23 *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, T be a fixed positive number, and $\mathscr{F}_t$, $0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathscr{F}$. Consider an adapted stochastic process X_t, $0 \leq t \leq T$. Assume that for all $0 \leq s \leq t \leq T$ and every non-negative, Borel-measurable function f, there is another Borel-measurable function g such that*

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = g(X_s),$$

then we say that the X process is a Markov process.

The most well-known stochastic process is called Brownian motion and it is named after its discoverer - Robert Brown. In 1827, the motion is discovered through the observation of microscopic pollen plant that is submerged in water. Many years after the discovery, the famous theoretical physicist Albert Einstein published a major paper regarding the movement/motion of pollen particles are actually caused by collision of individual water molecules (Einstein, 1905). Subsequently, Einstein's explanation has led Jean Perrin to experiment and verify the existence of atoms and molecules; that has won him a Nobel prize in Physics 1926 (Karlsson, 2001). We now give the definition of Brownian motion, its properties and the definition of quadratic variations.

Definition 1.24 *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t)$, $t \geq 0$ is a Brownian motion if for all $0 = t_0 < t_1 < ... < t_m$ the increments*

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), ..., W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$$

and

$$Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

Definition 1.25 *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space on which is defined on a W_t,*

$t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathscr{F}_t$, $t \geq 0$, satisfying:

1. **(Information accumulation)** *For $0 \leq s < t$, every set in $\mathscr{F}_s$ is also in $\mathscr{F}_t$. That is to say that there is at least as much information available at the later time $\mathscr{F}_t$ as compared to earlier time $\mathscr{F}_s$.*
2. **(Adaptivity)** *For each $t \geq 0$, the Brownian motion W_t at time t is $\mathscr{F}_t$-measurable. That is to say that the information available at time t is sufficient to evaluate the Brownian motion W_t at that time.*
3. **(Independence of future increments)** *For $0 \leq t < u$, the increment $W_u - W_t$ is independent of $\mathscr{F}_t$. That is to say any increment of Brownian motion after time t is independent of the information available at time t.*

Definition 1.26 *Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variance of f up to time T is*

$$\langle f, f \rangle(T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2,$$

where $\Pi = \{t_0, t_1, ..., t_n\}$, $0 = t_0 < t_1 < ... < t_n = T$, and $||\Pi|| = \max_{j=0,...,n-1} (t_{j+1} - t_j)$.

Theorem 1.1 *Let W be a Brownian motion. Then $\langle W, W \rangle(T) = T$ for all $T \geq 0$ almost surely.*

Definition 1.27 *(Nourdin, 2012) Let W_1 and W_2 be two independent classical Brownian motion, then the two-sided classical Brownian motion is defined as*

$$W_t = \begin{cases} W_t^1, & \text{if } t \geq 0, \\ W_{-t}^2, & \text{if } t < 0. \end{cases}$$

1.3 Stochastic Calculus

Stochastic calculus is mainly an extension of the ordinary calculus that deals with presence of non-zero quadratic variation in Brownian motion. Specifically, the famous Itô calculus (Itô, 1944) will be introduced in this subsection, we use the book by Shreve (2004) as the reference for all the definitions, lemmas, and theorems. The following work will be repeatedly used in this thesis.

Definition 1.28 *Let $X(t)$ be a stochastic process adapted to $\mathscr{F}(t)$ for $t \geq 0$ such that* *the square-integrability condition is satisfied as*

$$\mathbb{E}\left[\int_0^t X^2(u)du\right] < \infty$$

and W *be a classical Brownian motion, then the Itô integral is defined as*

$$I(t) = \int_0^t X(u)dW_u.$$

Theorem 1.2 *Let* T *be a positive constant and let* $X(t)$ *for* $0 \leq t \leq T$ *be an adapted stochastic process (same as Definition 1.28), then it satisfies the following properties:*

1. **(Continuity)** *The paths of* $I(t)$ *are continuous.*
2. **(Adaptivity)** *For each t,* $I(t)$ *is* $\mathscr{F}(t)$*-measurable.*
3. **(Linearity)** *Let* $X(u)$ *and* $Y(u)$ *be adapted stochastic process. If* $I(t) = \int_0^t X(u)dW_u$ *and* $J(t) = \int_0^t Y(u)dW_u$*, then* $I(t) \pm J(t) = \int_0^t (X(u) \pm Y(u))dW_u$*. In addition, for a constant c, the relationship* $cI(t) = \int_0^t cX(u)dW_u$ *is satisfied.*
4. **(Martingale)** $I(t)$ *is a martingale.*
5. **(Itô isometry)** $\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t X^2(u)du\right]$.
6. **(Quadratic variation)** $\langle I, I\rangle(t) = \int_0^t X^2(u)du$.

Lemma 1.1 *Let* $f(t,x)$ *be a function such that the partial derivatives* $f_t(t,x), f_x(t,x)$*, and* $f_{xx}(t,x)$ *are defined and continuous. Let* W_t *be a Brownian motion, then for every* $T \geq 0$*,*

$$f(T, W_T) = f(0, W_0) + \int_0^T f_t(t, W_t)dt + \int_0^T f_x(t, W_t)dW_t + \frac{1}{2}\int_0^T f_{xx}(t, W_t)dt.$$

Definition 1.29 *Let* W_t*,* $t \geq 0$ *be a Brownian motion, and let* $\mathscr{F}(t)$*,* $t \geq 0$*, be an associated filtration. An Itô process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t h(u)du + \int_0^t g(u)dW_u,$$

where $X(0)$ *is nonrandom and* $h(u)$ *and* $g(u)$ *are adapted stochastic processes such that* $\int_0^t |h(u)|du < \infty$ *and* $\mathbb{E}\left[\int_0^t g^2(u)du\right] < \infty$.

Lemma 1.2 *The quadratic variation of the Itô process (Definition 1.29) is*

$$\langle X,X\rangle(t) = \int_0^t g^2(u)du.$$

Definition 1.30 *Let $X(t)$, $t \geq 0$, be an adapted Itô process described in Definition 1.29. Furthermore, let $Y(t)$ be an adapted process, then the integral with respect to an Itô process can be defined as*

$$\int_0^t Y(u)dX(u) = \int_0^t Y(u)h(u)du + \int_0^t Y(u)g(u)dW_u.$$

Lemma 1.3 *Let $X(t)$, $t \geq 0$ be an adapted stochastic process (Definition 1.29), and let $f(t,x)$ be a function such that the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Let W_t be a Brownian motion, then for every $T \geq 0$,*

$$\begin{aligned} f(T,X(T)) &= f(0,X(0)) + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))dX(t) \\ &\quad + \frac{1}{2}\int_0^T f_{xx}(t,X(t))d\langle X,X\rangle(t) \\ &= f(0,X(0)) + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))h(t)dt \\ &\quad + \int_0^T f_x(t,X(t))g(t)dW_t + \frac{1}{2}\int_0^T f_{xx}(t,X(t))g^2(t)dt. \end{aligned}$$

Theorem 1.3 *Let W_s, $s \geq 0$, be a Brownian motion, and let $X(s)$ be a nonrandom function of time. Then, for each $t \geq 0$, the Itô integral $I(t) = \int_0^t X(s)dW_s$ is normally distributed with an expected value of zero and variance of $\int_0^t X^2(s)ds$.*

Lemma 1.4 *Let $f(t,x,y)$ be a function whose partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy} are defined and are continuous. Furthermore, let $X(t)$ and $Y(t)$ be adapted stochastic processes defined in Definition 1.29, then the two-dimensional Itô's Lemma in differential form is*

$$\begin{aligned} &df(t,X(t),Y(t)) \\ &\quad = f_t(t,X(t),Y(t))dt + f_x(t,X(t),Y(t))dX(t) + f_y(t,X(t),Y(t))dY(t) \\ &\qquad + \frac{1}{2}f_{xx}(t,X(t),Y(t))dX(t)dX(t) + f_{xy}(t,X(t),Y(t))dX(t)dY(t) \\ &\qquad + \frac{1}{2}f_{yy}(t,X(t),Y(t))dY(t)dY(t). \end{aligned}$$

Lemma 1.5 Itô's product rule *Let $X(t)$ and $Y(t)$ be adapted stochastic processes defined in Definition 1.29, then the Itô's product rule in differential form is*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t).$$

1.4 Fractional Calculus and Some Useful Functions

In the late 17th century, the classical infinitesimal calculus developed independently by Isaac Newton and Gottfried Wilhelm Leibniz has been widely adopted and have advanced many science-related fields. In the light of questions by L'Hopital to Leibniz such as "what is the derivative of order $\sqrt{2}$ or $1/3$ of a function", fractional calculus was defined and studied extensively. We will first describe several definitions regarding the indicator, Gamma, Beta, and Mittag-Leffler functions. The definitions will be used to define the rough Heston model and derive several methods in Chapter 3 and 4. Then, we will focus on the most famous type of fractional integral and derivative among the many definitions which is the Riemann-Liouville type. They will be used in the rough Heston model's characteristic function.

Definition 1.31 *(Hirsa & Neftci, 2013) Let A be any event, the indicator function is defined as*

$$\mathbb{1}_A = \begin{cases} 1, & \text{if event } A \text{ occurs}, \\ 0, & \text{otherwise}. \end{cases}$$

The Definitions 1.32 to 1.37 were taken from book by Baleanu et al. (2012).

Definition 1.32 *Let $x \in \mathbb{C}$, the Euler Gamma function (or more frequently known as just the Gamma function) is defined by the Euler integral of the second kind as*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{for } \Re(x) > 0,$$

where $\Re(\cdot)$ denotes the real part of the solution. Furthermore, the relationship of reduction formula can be established as

$$\Gamma(x+1) = x\Gamma(x).$$

Definition 1.33 *Let $x, y \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$, the Beta function is defined through Euler integral of the first kind as*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(\cdot)$ is the Gamma function described in Definition 1.32.

Definition 1.34 *Let $x \in \mathbb{C}$ and $\Re(\alpha) > 0$, the one-parameter Mittag-Leffler function is defined as*

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)},$$

where $\Gamma(\cdot)$ is the Gamma function described in Definition 1.32.

Definition 1.35 *Let $x, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$, the two-parameter (generalised) Mittag-Leffler function is defined as*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)},$$

where $\Gamma(\cdot)$ is the Gamma function described in Definition 1.32.

Definition 1.36 *Let $a \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ such that $\Re(\alpha) > 0$, the Riemann-Liouville fractional integral is defined as*

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

where $\Gamma(\cdot)$ is the Gamma function described in Definition 1.32. Furthermore, if the value a is not specified in the Riemann-Liouville integral, then let $a = 0$ and omit the value a in the fractional integral, i.e.

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt.$$

Definition 1.37 *Let $a \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ such that $\Re(\alpha) \geq 0$, the Riemann-Liouville fractional derivative is defined as*

$$D_a^{\alpha} f(x) = \frac{d^n}{dx^n} I_a^{n-\alpha} f(x),$$

where $n = \lceil \Re(\alpha) \rceil$ such that $\lceil \cdot \rceil$ denotes the ceiling of the argument and $\frac{d^n}{dx^n}$ is the n-th order derivative. Furthermore, if the value a is not specified in the Riemann-Liouville derivative, then let $a = 0$ and omit the value a in the fractional derivative and fractional integral, i.e.

$$D^{\alpha} f(x) = \frac{d^n}{dx^n} I^{n-\alpha} f(x).$$

Specifically, if $\alpha = n \in \mathbb{N}_0$, then

$$D^0 f(x) = f(x)$$

and

$$D^{\alpha} f(x) = D^{n} f(x) = \frac{d^{n}}{dx^{n}} f(x).$$

Next, we introduce a proposition and a lemma that will be used in Chapter 3.

Proposition 1.1 *(Khan et al., 2013; Agarwal, 1953) The m-th derivative of the generalised Mittag-Leffler function (Definition 1.35) is*

$$E^{(m)}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(n+m)!z^{n}}{n!\Gamma(\alpha(n+m)+\beta)}.$$

Lemma 1.6 *(Podlubny, 1997) Suppose that we have the function:*

$$\varepsilon_m(t,a,\alpha,\beta) = t^{\alpha m+\beta-1} E^{(m)}_{\alpha,\beta}(\pm a t^{\alpha}),$$

where $E^{(m)}_{\alpha,\beta}(\cdot)$ is defined in Proposition 1.1. Then, the Laplace transform of the function $\varepsilon_m(t,a,\alpha,\beta)$ is computed as

$$\begin{aligned}\mathscr{L}\{\varepsilon_m(t,a,\alpha,\beta)\} &= \int_0^{\infty} e^{-st} \varepsilon_m(t,a,\alpha,\beta)dt \\ &= \frac{m! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{m+1}}, \qquad (Re(s) > |a|^{1/\alpha})\end{aligned}$$

and the inverse Laplace transform relationship is as follows:

$$\mathscr{L}^{-1}\left[\frac{m! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{m+1}}\right] = t^{\alpha m+\beta-1} E^{(m)}_{\alpha,\beta}(\pm a t^{\alpha}).$$

The next three definitions will used in the numerical experiment of Chapter 2 to evaluate performance of certain methods.

Definition 1.38 *Let $\alpha = H + 0.5$ where the $H \in (0, 0.5)$ is the Hurst parameter, $f(x)$ be the exact solution to a fractional differential equation $D^{\alpha} f(x)$, and $\hat{f}(x)$ be the approximated solution of the same fractional differential equation. The squared Euclidean distance between the real part of the fractional differential equation with the exact solution $f(x)$ and the approximated solution $\hat{f}(x)$ is defined as*

$$\varepsilon^{H}_{Re} = \frac{1}{N} \sqrt{\sum_{n=1}^{N} \Re\left(D^{\alpha} \hat{f}(x_n) - D^{\alpha} f(x_n)\right)^{2}}$$

whereas for the squared Euclidean norm distance for the imaginary part of the fractional differential equation with the exact solution and fractional differential equa-

tion with the approximated solution is defined as

$$\varepsilon_{Im}^{H} = \frac{1}{N}\sqrt{\sum_{n=1}^{N} \Im\left(D^{\alpha}\hat{f}(x_n) - D^{\alpha} f(x_n)\right)^2}.$$

Definition 1.39 *Let $\alpha = H + 0.5$ where the $H \in (0, 0.5)$ is the Hurst parameter, $f(x)$ be the exact solution to a fractional differential equation $D^{\alpha} f(x)$, and $\hat{f}(x)$ be the approximated solution of the same fractional differential equation. The overall percentage error for the real part of the fractional differential equation with the approximated solution is defined as*

$$\hat{\varepsilon}_{Re}^{H} = \frac{1}{N}\sum_{n=1}^{N}\left|\frac{\Re\left(D^{\alpha}\hat{f}(x_n) - D^{\alpha} f(x_n)\right)}{\Re(D^{\alpha} f(x_n))}\right|,$$

whereas for the percentage error for the imaginary part of the fractional differential equation with the approximated solution is defined as

$$\hat{\varepsilon}_{Im}^{H} = \frac{1}{N}\sum_{n=1}^{N}\left|\frac{\Im\left(D^{\alpha}\hat{f}(x_n) - D^{\alpha} f(x_n)\right)}{\Im(D^{\alpha} f(x_n))}\right|.$$

Definition 1.40 *The overall error computation for the exact and imaginary part of the fractional differential equation $D^{\alpha} f(x)$ across different Hurst parameter H are defined as*

$$\chi_{Re} = \frac{1}{50}\sum_{j=1}^{50} \varepsilon_{Re}^{\min(j/100, 0.499)}, \qquad \chi_{Im} = \frac{1}{50}\sum_{j=1}^{50} \varepsilon_{Im}^{\min(j/100, 0.499)},$$

where ε_{Re}^{H} and ε_{Im}^{H} are described in Definition 1.38. The overall percentage error computation for the real and imaginary part of fractional differential across different Hurst parameter H are defined as

$$\Psi_{Re} = \frac{1}{50}\sum_{j=1}^{50} \hat{\varepsilon}_{Re}^{\min(j/100, 0.499)}, \qquad \Psi_{Im} = \frac{1}{50}\sum_{j=1}^{50} \hat{\varepsilon}_{Im}^{\min(j/100, 0.499)},$$

where $\hat{\varepsilon}_{Re}^{H}$ and $\hat{\varepsilon}_{Im}^{H}$ are described in Definition 1.39. Note that the overall error and overall percentage error computations will be considered for Hurst parameter $H = 0.01, 0.02, ..., 0.499$ or $\alpha = 0.51, 0.52, ..., 0.999$.

Definitions 1.41 and 1.42 are introduced next, the definitions will be used in the numerical experiment of Chapter 3.

Definition 1.41 *Let $\mathscr{T} = \{t_0, t_1, ...t_N\}$ where $t_i = T(i/N)$ for $i = 0, 1, ..., N$, $f(t)$ be the exact solution to a fractional differential equation $D^\alpha f(t)$, and $\hat{f}(t)$ be the approximated solution of the same fractional differential equation. The maximum absolute error between the imaginary part of the fractional differential equation with the exact solution $f(t)$ and the approximated solution $\hat{f}(t)$ is denoted as*

$$\mathscr{M}(H) = \max_{t \in \mathscr{T}} |\Im(D^\alpha f(t) - D^\alpha \hat{f}(t))|,$$

where $\Im(\cdot)$ refers to argument's imaginary part and $\alpha = H + 0.5$ where $H \in (0, 0.5)$ is the Hurst parameter.

Definition 1.42 *Let $f(t)$ be the exact solution to a fractional differential equation $D^\alpha f(t)$ and $\hat{f}(t)$ be the approximated solution of the same fractional differential equation. We define the absolute error between the imaginary part of the fractional differential equation with the exact solution $f(t)$ and the approximated solution $\hat{f}(t)$ as*

$$\mathscr{E}(t) = |\Im(D^\alpha f(t) - D^\alpha \hat{f}(t))|,$$

where $\Im(\cdot)$ refers to argument's imaginary part.

1.5 Fractional Brownian Motion

The generalisation of the Brownian motion is known as fractional Brownian motion (fBm). A main distinction between the classical Brownian motion and fBm is that the fBm's increments are not required to be independent, i.e. the covariance of fBm on its current increment can have positive, negative or zero correlation on its past self. The process is first introduced in Mandelbrot & Van Ness (1968). As fBm is similar to rough volatility models (see Chapter 2), we give several definitions related to fBm as follows:

Definition 1.43 *(Shreve, 2004) A Gaussian process $X(t)$, $t \geq 0$, is a stochastic process that has the property that, for arbitrary times $0 < t_1 < t_2 < ... < t_n$, the random variables $X(t_1), X(t_2), ..., X(t_n)$ are jointly normally distributed. Furthermore, if $\mathbb{E}[X(t)] = 0$ for all $t \geq 0$, then it is also a centred Gaussian process.*

Definition 1.44 *(Biagini et al., 2008) Let H be a constant belonging to $(0, 1)$. A fractional Brownian motion $B^H(t)_{t \geq 0}$ of Hurst parameter H is a continuous and centred Gaussian process with covariance function*

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

The rest of the propositions in this subsection can be found in Nourdin (2012).

Proposition 1.2 *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1]$. If $H = \frac{1}{2}$, then fractional Brownian motion is the same as a classical Brownian motion.*

Proposition 1.3 *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then:*

1. **(Self-similarity)** *For all $a > 0$, $\left(a^{-H} B^H_{at}\right)_{t \geq 0} \overset{law}{=} \left(B^H_t\right)_{t \geq 0}$.*

2. **(Stationary of increments)** *For all $h > 0$, $\left(B^H_{t+h} - B^H_h\right)_{t \geq 0} \overset{law}{=} \left(B^H_t\right)_{t \geq 0}$.*

3. **(Time inversion)** $\left(t^{2H} B^H_{1/t}\right)_{t>0} \overset{law}{=} \left(B^H_t\right)_{t>0}$.

Although there exist few stochastic representations of fractional Brownian motion process as Wiener integral (e.g., three integral representations in Nourdin, 2012), we will give the most widely known stochastic integral representation for the fBm.

Proposition 1.4 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, set*

$$c_H = \sqrt{\frac{1}{2H} + \int_0^\infty \left((1+u)^{H-0.5} - u^{H-0.5}\right)^2 du}.$$

Then, the fractional Brownian motion process of Hurst parameter H, $B^H = (B^H_t)_{t \geq 0}$ is defined as

$$B^H_t = \frac{1}{c_H} \left(\int_{-\infty}^0 \left((t-u)^{H-0.5} - (-u)^{H-0.5} \right) dW_u + \int_0^t (t-u)^{H-0.5} dW_u \right),$$

where $W = (W_t)_{t \in R}$ is a two-sided classical Brownian motion.

Remark 1.2 *Unlike classical Brownian motion, fBm is extremely hard to simulate due to its path dependent feature, and the common method to simulate fBm is the Cholesky decomposition method (for a review, see Dieker, 2004). Due to the recent findings (Gatheral et al., 2018), rough volatility models constructed using the features of fBm are becoming increasingly important to model stock and volatility process accurately. Specifically, in this study, we use a rough volatility model called rough Heston model to price option. We will discuss some problems related to rough Heston model and its option pricing method in Subsection 1.8.*

1.6 Option Pricing models

This section will specifically deal with European option models. The questions "how do we accurately price European options such that it reflects the prices we observe in the market" and "what factors do we have to consider in order to reflect the market price into the model" have been frequently asked for the past 50 years. A lot of progression on option pricing theory has been made. Local and stochastic volatility model are the two major type of models in option pricing theory. We will start this subsection by stating the definitions of local and stochastic volatility models to let readers understand the differences. The following definitions are taken from Gatheral (2011) unless specified otherwise.

Definition 1.45 *Let $\sigma(S_t, t; S_0)$ be a nonrandom function that depends on S_t, t, and S_0 (it is also known as local volatility function when consistent with the current European option prices). Furthermore, let $\mu(t)$ be a deterministic nonrandom function that depends on time t, the local volatility model is defined as*

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(S_t, t; S_0)dW_t,$$

where W is a one-sided classical Brownian motion.

Definition 1.46 *Let σ be a deterministic function such that it depends only on V_t, μ be a deterministic function such that it depends only on t. Then, the stochastic volatility model with stock price S and a stochastic process V is defined as*

$$dS_t = \mu(t)S_t dt + \sigma(V_t)S_t dW_t,$$

where W is a one-sided classical Brownian motion (Definition 1.24).

Between the local and stochastic volatility model, their main difference is the fit of implied volatility smile (mentioned in Definition 1.49 and Remark 1.5) of the models for short and long maturity-dated options. In particular, the local volatility model's implied volatility fits better in the shorter maturity option, whereas the stochastic volatility model's implied volatility is somewhat consistent in the longer maturity option (Hagan et al., 2002). See Dupire (1994) and Derman & Kani (1994) for more related discussion on local volatility models. There have been many stochastic volatility models proposed, e.g. the SABR volatility model (Hagan et al., 2002) which is a popular interest rate model that possesses the feature of reproducing volatility smile effect and the GARCH model (Engle, 1982) is a widely used stochastic volatility models with many variants (Montgomery et al., 2015).

Normally, it's preferred for the practitioners to use the local-stochastic volatility model rather than the individual local or stochastic volatility model. We would like to briefly mention about jump-diffusion models before moving on to the Black-Scholes model's discussion, i.e. the jump-diffusion models introduced by Kou & Wang (2004); Kou (2002); Merton (1976) are also extremely popular in option pricing theory because the model possesses diffusion process (seen during non-extreme market condition) and jump effects (when related news of the stock arrives).

We give the most basic definition of geometric Brownian motion as follows:

Definition 1.47 *A stochastic process S is said to follow the geometric Brownian motion if it satisfies the following stochastic differential equation:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W is a one-sided classical Brownian motion, $\mu \in \mathbb{R}$ is the constant for percentage drift, and $\sigma > 0$ is the percentage volatility constant.

As the stock process is generally accepted as the geometric Brownian motion such as in Definition 1.47 (Black & Scholes, 1973; Shreve, 2004; Gatheral, 2011), it would also indicate that the stock process is following a log-normal distribution. The stock process can actually be transformed into a stochastic process that is well-known for its distribution, i.e. the normal distribution through the use of Itô's Lemma 1.3. As such, a short lemma regarding the stochastic process (log-stock process) is given as follows:

Lemma 1.7 *(Hull, 2003) Suppose that S_t follows the geometric Brownian motion described in Definition 1.47, then the log-stock price $X_t := \log(S_t)$ follows the following stochastic differential equation:*

$$dX_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t.$$

1.6.1 Black-Scholes model

The construction of Black-Scholes-Merton partial differential equation (PDE) and its option pricing formulas (Black & Scholes, 1973; Merton, 1973) are given in this subsection. Their revolutionary work includes the derivation of the Black-Scholes-Merton differential equation and its option pricing formula. We will refer the trio "Black-Scholes-Merton" as "Black-Scholes" for simplicity purposes.

Assumption 1.1 *(Hull, 2003) The following assumptions are used to derive the famous Black-Scholes differential equation:*

1. *Stock price, S is driven by geometric Brownian motion process (Definition 1.47).*

2. *Short selling (Definition 1.12) of securities with full use of proceeding is permissible.*

3. *No transaction costs or taxes.*

4. *All securities are perfectly divisible, i.e. the purchase of fractional shares is permissible.*

5. *Absent of dividend during the life of derivative.*

6. *Riskless arbitrage opportunities is not permitted.*

7. *Security trading is continuous.*

8. *Constant risk-free interest rate, r throughout all maturities.*

Next, we discuss the Black-Scholes PDE's derivation. The rest of the materials which include lemma, definitions, and theorem are taken from Shreve (2004). We can view $c(t,S_t)$ as the call option value with stock price S_t at time t.

Lemma 1.8 *Let Assumption 1.1 holds and $c(t,S_t)$ be a function that depends only on time t and the value S_t, then the differential $dc(t,S_t)$ is*

$$dc(t,S_t) = \left[\frac{\partial}{\partial t}c(t,S_t) + \mu S_t \frac{\partial}{\partial S_t}c(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2}c(t,S_t)\right] dt + \sigma S_t \frac{\partial}{\partial S_t}c(t,S_t)dW_t,$$

where S_t is the stock price at time t.

Lemma 1.9 *Let Assumption 1.1 holds and $c(t,x)$ be a function that depends only on time t and the value x, then for interest rate r, the differential on discounted option price $d(e^{-rt}c(t,S_t))$ is*

$$d(e^{-rt}c(t,S_t)) = e^{-rt}\left[-rc(t,S_t) + \frac{\partial}{\partial t}c(t,S_t) + \mu S_t \frac{\partial}{\partial S_t}c(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial S_t^2}c(t,S_t)\right] dt + e^{-rt}\sigma S_t \frac{\partial}{\partial S_t}c(t,S_t)dW_t.$$

Remark 1.3 *The goal of Lemma 1.9 is to show how the evolution process of present value of option value is affected by the risk-free component, movement of the stock, quadratic variation of the stock and the fluctuating component.*

Definition 1.48 *Let X_t denotes the portfolio at time t. A short option hedging involves the portfolio starting with an initial capital of X_0, then the investor will own Δ_t worth of stock S_t at time t, invest the rest at money market account with interest rate r such that the discounted value of X_t will agree or match the value a discounted option $c(t,S_t)$. The value Δ_t will be picked to agree with the value of option price $c(t,S_t)$ at time t, that is*

$$e^{-rt}X_t = e^{-rt}c(t,S_t),$$

where X_t is modelled by an Itô process as

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt.$$

Accordingly, the portfolio value movement at time t is governed by the stock movement with magnitude of Δ_t and the time movement with magnitude $r(X_t - \Delta_t S_t)$ (gaining interest r at the residue of the portfolio value after investing in the stock). The component Δ_t will be determined in the next theorem. Note that at time zero, the initial portfolio value is $X_0 = c(0,S_0)$.

Theorem 1.4 *Suppose that Assumption 1.1 holds and short option hedge (Definition 1.48) is implemented such that the portfolio movement is always zero, then we can obtain the Black-Scholes partial differential equation as*

$$rc(t,S_t) = \frac{\partial}{\partial t}c(t,S_t) + rS_t\frac{\partial}{\partial S_t}c(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2}{\partial S_t^2}c(t,S_t) \tag{1.1}$$

for $S_t > 0$ and $t \in [0,T)$ where T is option's maturity time and the Equation (1.1) satisfies the call option's terminal condition as

$$c(T,S_T) = (S_T - K)^+.$$

Remark 1.4 *The Black-Scholes partial differential differential equation (Theorem 1.4) does not have to hold at $t = T$, but the $c(t,S_t)$ has to be continuous as $t = T$ as the hedge will work at time T due to the function continuity prior to T (Shreve, 2004). Mathematically, suppose that we have an initial capital of $X(0) = c(0,S_0)$ and at time t, we purchase $\Delta(t) = \frac{\partial}{\partial S_t}c(t,S_t)$ amount of stock and invest the rest at risk-free rate r, what will happen is that as the option reaches its maturity ($t \to T$), the portfolio value $X(T) = c(T,S_T) = (S_T - K)^+$ regardless of which possible stock price paths it follows. In addition, we can view the Black-Scholes strategy in two ways (only mathematically possible): (1) we can replicate the effects of option without buying the option itself, (2) if we short a unit of call option, it is possible to hedge the movement of options using only stocks, i.e. the short option position plus the investment in stocks and accumulation of interest rate in money-making account will always be zero.*

Theorem 1.5 *Suppose that $S_t > 0$ is the current stock price at time t, $K > 0$ is the strike price, $r > 0$ is the risk-free interest rate, $T - t$ is the time to maturity where $t \in [0,T)$, and $\sigma > 0$ is the volatility constant, then the solution to the Black-Scholes partial differential equation with the terminal condition stated in Theorem 1.4 is*

$$c(t,S_t) = S_t N(d_+(S_t,T-t)) - Ke^{-r(T-t)} N(d_-(S_t,T-t)), \qquad (1.2)$$

where

$$d_\pm(S_t,T-t) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S_t}{K} + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right]$$

and $N(\cdot)$ is the cumulative standard normal distribution.

Definition 1.49 *Let $c(S_t,K,\sigma,r,T-t)$ denotes the Black-Scholes call option pricing formula in Equation (1.2), then we say that σ_{imp} is the implied volatility to the option price V_t when*

$$c(S_t,K,\sigma_{imp},r,T-t) = V_t.$$

Remark 1.5 *From Definition 1.49, we can notice that the implied volatility cannot be computed in closed-form manner, but it can actually be efficiently solved numerically using methods such as the bisection method since the call option price is non-decreasing on σ. Furthermore, the implied volatility smile is a plot (typically having a smile shape) that describes the change of implied volatility against the strike price K.*

We introduce a simple performance measure for the differences between the exact and the approximation for implied volatility next. The performance measure will be used in the numerical experiment in Chapter 3.

Definition 1.50 *Suppose that T is maturity time, H is the Hurst parameter, k is the log-strike $k = \log(K/S_0)$, $\sigma(H,T,k_i)$ is the exact implied volatility, and $\widehat{\sigma}(H,T,k_i)$ is the approximation for implied volatility. The implied volatility's average absolute error is defined as*

$$\mathscr{A}(H,T) = \frac{1}{N_s}\sum_{i=1}^{N_s} |\sigma(H,T,k_i) - \widehat{\sigma}(H,T,k_i)|,$$

where k_i is the log-strike price sampled in the range $k_i \in [-0.4, 0.4]$ and N_s is the sample size for the implied volatility.

1.6.2 Classical Heston model

Let us first mention that the stochastic volatility model - Heston model is introduced in Heston (1993). Furthermore, it is in fact preferable when compared to the Black-

Scholes and Hull-White model (Hull & White, 1990) for the following reasons:

1. Several known patterns of low-frequency data can be reproduced by classical Heston model, that includes the fat tails effect, possessing time-varying volatility and leverage effect (market volatility is negatively correlated to stock return)

2. Reasonable implied volatility surface generated by option price under classical Heston model.

3. Computation of the Heston model's characteristic function can be performed instantaneously; there exists several efficient numerical methods for pricing derivatives (Carr & Madan, 1999; Lewis, 2001, 2009).

We discuss the definition of Heston model, the PDE of call option on Heston dynamics, and the solution of squared-volatility in Heston model.

Definition 1.51 *(Heston, 1993; Gatheral, 2011) Let S_t be the stock price and $\sqrt{V_t}$ be the volatility of the stock price at time t. The Heston model is defined as*

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1$$

and

$$dV_t = \lambda(\theta - V_t)dt + \nu\sqrt{V_t} dW_t^2,$$

where μ is the mean rate of return of stock return, λ is the speed or magnitude of mean reversion of V_t to the mean squared-volatility θ, as well as ν is the magnitude of volatility movement (sometimes also known as volatility of the volatility). Furthermore, W_t^1 and W_t^2 are independent one-sided classical Brownian motion that is correlated by ρ such that

$$dW_t^1 dW_t^2 = \rho dt.$$

Theorem 1.6 *(Heston, 1993; Gatheral, 2011) Let $C_t := C(S_t, K, \mu, \lambda, \theta, \nu, \rho)$ denotes the value of call option price and it follows the Heston model dynamics described in Definition 1.51. From the Delta-Vega hedge argument, the partial differential equation that governs the movement of the option price is*

$$\begin{aligned} rC_t = \frac{\partial C_t}{\partial t} + rS_t \frac{\partial C_t}{\partial S_t} + \rho\nu V_t S_t \frac{\partial^2 C_t}{\partial V_t \partial S_t} + \frac{1}{2} V_t S_t^2 \frac{\partial^2 C}{\partial S^2} \\ + \frac{1}{2}\nu^2 V_t \frac{\partial^2 C_t}{\partial V_t^2} + \lambda(\theta - V_t)\frac{\partial C_t}{\partial V_t} \end{aligned}$$

for $t \in [0, T]$ where T is the time to maturity of the option and the terminal condition for call option is

$$C_T = (S_T - K)^+.$$

Remark 1.6 *Unlike the Black-Scholes model, closed-form solution is not available for the call option under the Heston model, instead, it possesses a semi-closed form solution using the Fourier transform method/characteristic function method (Gatheral, 2011).*

Lemma 1.10 *(Shreve, 2004) Let V_t be the Heston process described in Definition 1.51, the solution to the stochastic differential equation is*

$$V_t = V_0 e^{-\lambda t} + \theta(1 - e^{-\lambda t}) + \nu e^{-\lambda t} \int_0^t e^{\lambda u} \sqrt{V_u} dW_u^2.$$

where λ, θ, and ν are described in Definition 1.51

1.7 Characteristic function

In probability theory, the characteristic function plays an important role in studying random variables, this is because analytical properties of the random variables' characteristic functions often correspond to its probabilities properties (Tankov, 2003). In simpler terms, the random variable's characteristic function corresponds to its Fourier transformed distribution. Later in this chapter, the connection of characteristic function and option pricing will be discussed. Readers may see Tankov (2003) for the definitions of Lévy process and Lévy measure. The next lemma is famously known as Lévy-Khintchine representation.

Lemma 1.11 *(Tankov, 2003) Let X_T be a Lévy process, then the characteristic function of X_T has a representation*

$$\Phi_{X_T}(u) = \exp\left(iu\omega T - \frac{1}{2}u^2\sigma^2 T + T\int_{\mathbb{R}} \left[e^{iux} - 1 - iux\mathbb{1}_{|x|<1} \right] L(dx) \right),$$

where σ is the volatility of the Lévy process (such as the term σ in Definition 1.47), ω is the drift parameter, $\mathbb{1}$ is the indicator function, and L is the Lévy measure of X satisfying the property:

$$\int_{\mathbb{R}} (|x|^2 \wedge 1) L(dx) < \infty.$$

Remark 1.7 *The ω parameter in Lévy-Khintchine representation (Lemma 1.11) can be obtained under some assumptions such as the absence of interest rate and dividend, this leads to certain equality:*

$$\Phi_{X_T}(-i) = \mathbb{E}[e^{X_T}] = 1.$$

Lévy process and Lévy measure will not be further discussed in the subsequent chapters.

Definition 1.52 *(Tankov, 2003) Let X be a random variable such that the characteristic function of the random variable X is defined as*

$$\Phi_X(u) = \mathbb{E}[e^{iuX}],$$

where i is the imaginary unit.

Theorem 1.7 *(Gatheral, 2011) Let X_t be the log-stock price $X_t = \log(S_t)$ where S_t follows the geometric Brownian motion (Definition 1.47), then the characteristic function of X_T is*

$$\Phi_X(u) = \mathbb{E}\left[e^{iuX}\right] = e^{-\frac{1}{2}u\sigma^2 T(u+i)},$$

where i is the imaginary unit.

Remark 1.8 *While there is a characteristic function representation of Black-Scholes model, it is not used as often as there exists a closed-form option solution (Theorem 1.5).*

The next theorem is the characteristic function of the Heston model from Heston (1993) or similarly from Gatheral (2011), we have used another representation of the characteristic function (it leads to the same result in El Euch & Rosenbaum (2019) and El Euch et al. (2019)).

Theorem 1.8 *Let X_t be the log-stock price $X_t = \log(S_t)$ where S_t is governed by the Heston model (Definition 1.51), then the characteristic function of X_T is*

$$\Phi_{X_T}(a) = \exp\left(iaX_0 + \int_0^T \frac{\partial}{\partial u} h(a, T-u)\xi_0(u)du\right),$$

where $\xi_0(t) = \mathbb{E}_0[V_t]$ and the solution to the Riccati equation $h(a,t)$ is:

$$\frac{\partial}{\partial t} h(a,t) = \frac{1}{2}\nu^2 h^2(a,t) + i\rho\nu a h(a,t) - \frac{1}{2}a(a+i); \quad h(a,0) = 0.$$

The solution $h(a,t)$ is as follows:

$$h(a,t) = r_- \frac{1 - e^{-A\nu t}}{1 - \frac{r_-}{r_+} e^{-A\nu t}}$$

with the components A and $r_\pm$ being

$$A = \sqrt{-\rho^2 a^2 + a(a+i)}; \quad r_\pm = -\frac{1}{\nu}(i\rho a \pm A).$$

It can be observed that the characteristic function of a certain model is a form of Fourier transform, therefore we are required to use Fourier inversion method to inverse it back to option price, which we will introduce in next theorem. There are many different formulas that can compute for the option price through the Fourier inversion method, but we have chosen the Lewis' call option pricing formula from Lewis (2001), this is because it is more well-known than other Fourier inversion methods for option price (Lewis, 2009; Gatheral, 2011).

Theorem 1.9 *Suppose that the characteristic function of log-stock price X_T is known and denoted as $\Phi_{X_T}(u)$, then the call option price as a function of characteristic function is*

$$C(S,K,T) = Se^{-qT} - \frac{\sqrt{SK}}{\pi} e^{-(r+q)\frac{T}{2}} \int_0^\infty \Re \left[e^{-iuk} \Phi_{X_T}\left(u - \frac{i}{2}\right)\right] \frac{du}{u^2 + \frac{1}{4}},$$

where S is the stock price, K is the strike price, r is the risk-free interest rate, q is the dividend yield, T is the maturity time of the option, and $k := \log(K/S)$.

Remark 1.9 *A recent work by Baschetti et al. (2020) has noted that an alternative method called SINC approach (Cherubini et al., 2010) can reach a desired accuracy in option pricing under rough Heston model substantially faster than the Lewis' method (Theorem 1.9) and Carr-Madan's method (Carr & Madan, 1999).*

1.8 Problem Statements and Objectives of the Thesis

Currently, there are several problems in option pricing theory that have been raised:

- Difficulty in computing rough Heston model's option price, i.e. the conventional computation method requires high computational cost (El Euch & Rosenbaum, 2019).
- Lack of accurate, efficient, and robust approximation methods to price option under rough Heston model (Gatheral & Radoičić, 2019; Abi Jaber & El Euch, 2019), e.g. the multipoint Padé approximant introduced by Gatheral & Radoičić (2019) performs well at low Hurst parameter (near zero), but performs poorly at higher Hurst parameter (near 0.5).

As such, the main objectives of this thesis are:

- Modify and improve an existing method for a better approximation on the fractional Riccati equation's solution, as the particular equation is the main contributing factor of the high computational cost when computing for the rough Heston model's option price.

- Develop an efficient option approximation formula for rough Heston model without involving the characteristic function of rough Heston model and Fourier inversion method.

1.9 Outline of the Thesis

Motivated by some of the unsolved problems in option pricing theory stated in Subsection 1.8, the rest of this thesis will be divided into five chapters that contain the literature review, advancement of the option pricing theory that relates to rough volatility models, and of course the conclusion.

In Chapter 2, it contains a lengthy review for the study of the rough volatility models, the focus of the literature review will be placed on rough Heston model and its related numerical methods. In particular, an existing approximation method called multipoint Padé approximation is reviewed, discussed, and compared against the conventional numerical method called fractional Adams method.

In Chapter 3, it continues from the work of Chapter 2 such that we have modified and improved the existing Padé approximant to develop three variants of higher-order Padé approximant with different construction and formalisation of the multipoint Padé approximation method. Before that, as the rough Heston model's characteristic function contains the fractional Riccati equation and it is hard to solve, we employ the Laplace-Adomian-Decomposition method for obtaining the equation's solution in terms of series expansion. Numerical experiment is provided to compare the improved Padé approximants against the existing Padé approximant on the quality of fractional Riccati equation's solution for different Hurst parameter H and to verify its effectiveness pricing options under rough Heston model. The S&P 500 option is also used as the performance benchmark.

In Chapter 4, based on the rough Heston model, its option price's approximation formula is derived by using an existing method called the decomposition method. The error bound is also provided. Based on the approximation formula, a second-order implied volatility approximation is proposed and discussed. Specifically, we have derived two short-time implied volatility approximation behaviours, e.g. the term structure of at-the-money implied volatility skew when maturity time is short. Similar to Chapter 3, the performance of the approximation formula is tested on S&P 500 option. In particular, two different scenarios which are the non-extreme and extreme market conditions are considered for the calibration of the approximation formula on the S&P 500 options.

In Chapter 5, a conclusion is given, where it reiterates the research problems and summarises the main contributions of this thesis. Some interesting recommendations such as efficient computation and possible option pricing methods on the rough Heston model are also suggested as future research.

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