UNIVERSITI PUTRA MALAYSIA

INTERVAL-VALUED FUZZY SOFT TOPOLOGY AND ITS APPLICATIONS IN GROUP DECISION-MAKING PROBLEMS

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By

MABRUKA ALI JUMA ALTWER

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2022

DEDICATIONS

To my beloved ones;
my mother, my husband and my kids

Abstract of thesis presented to the Senate of University Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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April 2022

Chairman : Prof Adem Kılıçman, PhD
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Interval-valued fuzzy soft sets are an extension of fuzzy soft sets, which are used in decision-making to indicate insufficient evaluation, uncertainty, and vagueness. Lower membership degree and upper membership degree are two types of information considered by interval-valued fuzzy soft sets. In the literature, there are various interval-valued fuzzy soft set-based decision-making algorithms. However, these algorithms are unable to overcome the issue of comparable alternatives, and as a result, they might well be ignored due to a lack of a comprehensive model. In addition, generalizing preorder and equivalence of interval-valued fuzzy soft sets have been proposed. This generalization shows a deeper insight into the decision-making processed based on preference relationship. In this thesis, we develop two multi algorithms based on the interval-valued fuzzy soft topology to overcome different situations in decision-making problems.

In the first step, we present the interval-valued fuzzy soft topology concept as the basic framework of this work and we study some topological properties. This includes interior, closure, and continuity. Quasi-separation axioms in an interval-valued fuzzy soft topology, known as q-T_i spaces for $i = 0, 1, 2, 3, 4$, together with several of their basic properties are investigated.

In the second phase, we consider two crisp topological spaces, known as a lower topology induced by the interval-valued fuzzy soft topology $(IVFST)$, denoted as $\tau^{l}_{e,\beta}$ and an upper topology induced by the interval-valued fuzzy soft topology $(IVFST)$, denoted as $\tau^{u}_{e,\alpha}$. Some properties of these topologies are also studied. The induced topologies and quasi-separation axioms in interval-valued fuzzy soft topology are discussed.

In the third phase, we introduce two preorder relations and two equivalence relations over X for the two topological structures $\tau^{l}_{e,\beta}$ and $\tau^{u}_{e,\alpha}$. We also present some properties of these preorder and equivalence relations, and links between them are studied. The links between two preorder and equivalence relations and interval-valued fuzzy soft quasi-separation axioms are studied.

In the application phase of this thesis, we provide a representation of the results acquired in the previous steps in order to compute and define various algorithms that assist group decision-making using interval-valued fuzzy soft sets. The weighted interval-valued fuzzy soft set presented is applied to solve group decision-making using interval-valued fuzzy soft sets.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

TOPOLOGI LEMBUT KABUR BERNILAI SELANG DAN APLIKASINYA DALAM MASALAH MEMBUAT KEPUTUSAN BERKUMPULAN

Oleh

ALTWER MABRUKA ALI JUMA

April 2022

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Set lembut kabur bernilai selang ialah lanjutan set lembut kabur, yang digunakan dalam membuat keputusan untuk menunjukkan penilaian yang tidak mencukupi, ketidak pastian dan kekaburan. Darjah keanggotaan bawah dan darjah keanggotaan atas merupakan dua jenis maklumat yang dipertimbangkan oleh set lembut kabur bernilai selang. Dalam sorotan literatur, terdapat pelbagai algoritma yang membuat keputusan berasaskan set lembut kabur bernilai selang. Walau bagaimanapun, algoritma ini tidak dapat mengatasi isu alternatif yang setanding, dan akibatnya, ia mungkin diabaikan kerana kekurangan model yang komprehensif. Di samping itu, membuat generalisasi tertib awalan dan kesetaraan set lembut kabur bernilai selang telah dicadangkan. Generalisasi ini menunjukkan gambaran yang lebih mendalam tentang proses membuat keputusan yang diutamakan ke atas hubungan pilihan. Dalam tesis ini, kami membangunkan dua algoritma berbilang berdasarkan topologi lembut kabur bernilai selang untuk mengatasi situasi yang berbeza dalam permasalahan membuat keputusan.

Dalam langkah pertama, kami mengemukakan konsep topologi lembut kabur bernilai selang sebagai rangka kerja asas kajian ini dan kami menyelidiki beberapa sifat topologi. Ianya termasuk pedalaman, penutupan, dan keselanjaran. Aksiom pemisahan kuasi dalam topologi lembut kabur bernilai selang, yang dipanggil ruang $q-T_i$ bagi $i = 0,1,2,3,4$, bersama dengan beberapa sifat asasnya juga dikaji.

Dalam fasa kedua, kami mempertimbangkan dua ruang topologi krisp, dikenali sebagai topologi bawahan teraruh oleh topologi IVFS, ditulis sebagai $\tau^l_{e,\beta}$ dan topologi atasan teraruh oleh topologi IVFS, ditulis sebagai $\tau^u_{e,\alpha}$. Beberapa ciri-ciri

topologi tersebut dikaji. Topologi terauh $\tau^{l}_{e,\beta}$ dan $\tau^{u}_{e,\alpha}$ serta aksiom pemisahan kuasi dalam topologi lembut kabur bernilai selang dibincangkan.

Dalam fasa ketiga, kami memperkenalkan dua hubungan tertib awalan dan dua hubungan kesetaraan ke atas X bagi kedua-dua struktur topologi $\tau^{l}_{e,\beta}$ dan $\tau^{u}_{e,\alpha}$. Kami juga mengemukakan beberapa ciri-ciri hubungan tertib awalan dan kesetaraan, berserta perkaitan di antara mereka. Perkaitan di antara dua hubungan tertib awalan dan kesetaraan serta aksiom pemisahan kuasi dalam topologi lembut kabur bernilai selang telah dikaji.

Dalam fasa aplikasi tesis ini, kami menyediakan perwakilan keputusan yang diperoleh dalam langkah-langkah sebelumnya untuk mengira dan mentakrifkan pelbagai algoritma yang membantu membuat keputusan berkumpulan menggunakan set lembut kabur bernilai selang. Set lembut kabur bernilai selang berpemberat yang dibentangkan digunakan untuk menyelesaikan pembuatan keputusan berkumpulan menggunakan set lembut kabur bernilai selang.

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I owe a debt of gratitude to my loving mother and my father's spirit for my spiritual upbringing and support during my life. A huge thank you may not be adequate for their support, tolerance, understanding, as well as prayers, which have been important in my accomplishment.

Finally, words fail me to describe my gratitude to my spouse, whose commitment, love, and unwavering faith in me have lifted a heavy burden off my shoulders, as well as providing constant encouragement and inspiration to study and support during my life.

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

FS	Fuzzy Sets
SS	Soft Sets
IVF	Interval-Valued Fuzzy
$IVFS$	Interval-Valued Fuzzy Soft
$IVFSS$	Interval-Valued Fuzzy Soft Sets
GDM	Group Decision-Making
$IVFT$	Interval-Valued Fuzzy Topology
$IVFST$	Interval-Valued Fuzzy Soft Topology
(X,τ)	Topological Space
(X,E,τ)	Interval-Valued Fuzzy Soft Topological Space
$IVFSN$	Interval-Valued Fuzzy Soft Neighborhood
$L.C.S^f_\beta(e)$	β-Lower Crisp Set
$U.C.S^f_\alpha(e)$	α-Upper Crisp Set
$\tau^l_{e,\beta}$	lower topology induced by the interval-valued fuzzy soft topology
$\tau^u_{e,\alpha}$	upper topology induced by the interval-valued fuzzy soft topology
$\mathscr{I}\mathscr{V}\mathscr{F}(X)$	The set of all interval-valued fuzzy soft sets over X

CHAPTER 1

INTRODUCTION

1.1 Background

In this section, we provide some background about various pertinent topics of this study.

The fuzzy set(FS) theory incorporates impreciseness of data and evaluations by imputing the degrees to which objects belong to a set. Its appearance induced the rise of several related theories, which codify subjectivity, imprecision, uncertainty or roughness of evaluations. Their rationale is to produce new and more flexible methodologies to realistically model various concrete decision problems.

In 1965, Zadeh proposed (FS) theory as a mathematical approach for describing fuzzy phenomena in mathematics. Zadeh defined the concept of a (FS) theory in the following way:

Definition 1.1 *Zadeh (1965) Let X be a universe set. A fuzzy set μ on X is defined by a membership function $\mu : X \to [0,1]$, where the value $\mu(x)$ denotes the membership degree of $x \in X$ belong to the fuzzy set μ.*

The fuzzy set μ is also denoted as follows:

$$\mu = \{(x, \mu(x) : x \in X\}.$$

Some theories describe decision parameters, such as (FS) theory, (SS) theory, and rough set theory. In every situation, we use one of them. For example, in fuzzy set theory, we have an exact value for membership degree all the time. However, sometimes in the data collection step, we tend to miss data [unknown/ incomplete data], or sometimes we are not sure about the membership function such as e^x or $\frac{1}{x}$.

1.1.1 Soft Set Theory and Interval-Valued Fuzzy Soft Set Theory

Soft set (SS) theory was developed by Molodtsov (1999), which provided a broad mathematical framework for handling uncertain, vaguely specified objects with defined features. Soft sets are based on the concept of parameterizations, which have breached the boundaries of membership function.

Definition 1.2 *Molodtsov (1999) A pair (F,E) is known as a soft set (SS) over an initial universe set X, where F resembles a mapping stated by $F : E \rightarrow P(X)$, while $P(X)$ denotes the power set of X, while E indicates the set of parameters.*

In other words, a soft set over X is parameterized family of subsets X. Moreover, E is referred to as the soft parameter set of (F,E). Here, $F(e)$ may be considered for all $e \in E$ as a collection of e-approximate elements (F,E).

The fuzzy set theory and rough set theory are all special cases of the soft set theory, according to Aktaş and Cagman (2007). Besides that, Yao (1998) investigates the connections and differences between fuzzy set theory and rough set theory. Moreover, many authors discussed the link between fuzzy set theory and soft set theory. For example, Molodtsov (1999) proposed that the fuzzy set theory is a special case of the soft set theory. Meanwhile, Maji et al. (2002) were the first to investigate hybrid structures involving soft sets and fuzzy sets, proposing the notion of fuzzy soft sets as a fuzzy expansion of classical soft sets and discussing some of their essential features. Furthermore, Alcantud (2016b) examined several fuzzy set and soft set transformation relations. He established that every fuzzy set is a particular case of a soft set and that every soft set on a finite field U is a fuzzy set. Finally, Liu et al. (2019) compared fuzzy set and soft set theories from the perspective of transformation. The authors proved that every fuzzy set theory could resemble a soft set theory and showed that any soft set theory could be regarded as a fuzzy set theory.

Definition 1.3 *Maji et al. (2001) A pair (f,E) is called an FS set over X if the mapping f is given by $f : E \rightarrow [0,1]^X$ for all $e \in E$ and $x \in X$, with membership function $f_e = X \rightarrow [0,1]$.*

Many problems have been effectively solved using the concept of fuzzy soft sets(Cagman et al. (2011a), Cagman et al. (2010), Kong et al. (2009), Tripathy et al. (2016), Jun et al. (2010), Roy and Maji (2007), Çelik and Yamak (2013), Feng et al. (2010a), Alcantud (2016a), Alcantud and Mathew (2017),Aktaş and Cagman (2016)).

There are many extended soft sets, for example, rough soft set (Feng et al. (2010b)), interval-valued fuzzy soft set (Son (2007), Yang et al. (2009)), intuitionistic fuzzy soft set (Xu et al. (2010b)) and multi-fuzzy soft set (Yang et al. (2013)).

For interval-valued fuzzy set theory, which was first proposed by Son (2007) combines the soft set theory and interval-valued fuzzy set. It has been considered a more effective technique of depicting ambiguous information in a larger range.

Definition 1.4 *Son (2007) A pair (f,E) is called an $IVFS$ set over X if the mapping f is given by $f : E \rightarrow [0,1]^X$, where for any $e \in E$ and $x \in X$, we have $f(e)(x) = [f^-(e)(x), f^+(e)(x)]$.*

The $IVFS$ set is quickly becoming an interesting research topic due to its advantages as presented by (Feng et al. (2018)). This includes algebraic structure(Liu et al. (2014a), Liu et al. (2014b)), matrix theory (Rajarajeswari and Dhanalakshmi (2014)), parameter reduction (Ma et al. (2013), Ma and Qin (2020)), information measure (Dong et al. (2021), Jiang et al. (2013), Peng and Yang (2015)) and decision-making problems (Yang et al. (2009), Feng et al. (2010c), Yuan and Hu (2012), Xiao et al. (2013), Saeed (2016), Yang and Peng (2017), Chen and Zou (2017), Peng and Garg (2018)).

1.1.2 Topology and Interval- Valued Fuzzy Topology

Topology is a branch of mathematics studying spatial space structure, and it is the foundation for most pure and applied mathematics.

Definition 1.5 *Mondal and Samanta (1999) Topology over a set X is a collection τ of subsets of X, called the open sets, satisfying*

1- $\emptyset$ and X belong to τ,

2- Any union of elements of τ belongs to τ,

3- Any finite intersection of elements of τ belong to τ.

Then, τ is called topology over X, and the pair (X,τ) is called topological space. Similar to the ordinary topologies, the indiscrete topology over X contains only $\emptyset$, and X, comprises entire topological space. Every member of τ is referred to as an open set in X. Moreover, a closed set is the complement of an open set.

Definition 1.6 *(Mondal and Samnta,1999) The interval-valued fuzzy point (IVF-point)$\tilde{x}$ with support $x \in X$ as well as lower and upper λ^-,λ^+, respectively, for any interval $[\lambda^-,\lambda^+] \subseteq [0,1]$ and for each $y \in X$ is defined by*

$$\tilde{x}(y) = \begin{cases} [\lambda^-,\lambda^+], & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.7 *(Mondal and Samanta, 1999)Let (X,τ) be an IVF topological space and $\tilde{x}$ denote an interval-valued fuzzy point (IVF-point) having support x and lower and upper values λ^-,λ^+, respectively. The IVF set g is called the interval-valued fuzzy neighborhood $(IVFN)$ of IVF-point $\tilde{x}$ if there exist the IVF-open set f in X such that $x\tilde{\in} f \leq g$.*

Proposition 1.1 *(Mondal and Samanta, 1999)*

1. *If f is an IVFN of the IVF-point $\tilde{x}$ and $f \tilde{\leq} h$, then h is also IVFN of $\tilde{x}$.*

2. *If f and g are two IVFN of the IVF-point $\tilde{x}$, the $f \tilde{\wedge} g$ is also IVFN of $\tilde{x}$.*

3. *If f is an IVFN of the IVF-point $\tilde{x}_1$ and g is an IVFN of the IVF-point $\tilde{x}_2$, then $f \tilde{\vee} g$ is also an IVFN of $\tilde{x}_1$ and $\tilde{x}_2$.*

4. *If f is an IVFN of the IVF-point $\tilde{x}$, then there exists IVFN g of $\tilde{x}$ such that $g \tilde{\leq} f$ and g is an IVFN of the IVF-point $\tilde{y}$ for all $\tilde{y} \in g$.*

Definition 1.8 *(Mondal and Samanta, 1999) Let (X, τ) be an interval-valued fuzzy topological space and suppose f is an IVF. Then*

1 *The IVF-closure of f is the IVF set defined by*

$$CL(f) = \tilde{\wedge}\{g : g \text{ is IVF-closed set and } g \tilde{\leq} f\}$$

2 *The IVF-interior of all IVF-open subsets of f is the IVF set defined by*

$$Int(f) = \tilde{\vee}\{h : h \text{ is IVF-open set and } f \tilde{\leq} h\}$$

Definition 1.9 *(Mondal and Samanta, 1999) Suppose that (X, τ_1) and (Y, τ_2) are two IVFS topological spaces. Then, a mapping $\Phi : (X, \tau_1) \to (Y, \tau_2)$ is said to be:*

1 *An IVF-continuous map if $\Phi^{-1}(g)$ is an IVF-open set of X, for all IVF-open set g of Y.*

2 *An IVF-open map if $\Phi(f)$ is an IVF-open set of Y, for each IVF-open set g of X.*

3 *An IVF-homeomorphism map if Φ is IVF-bijective and Φ^{-1} is IVF-continuous.*

Definition 1.10 *(Kharal and Ahmad, 2011) Let $\mathscr{S}(X_1, E_1)$ and $\mathscr{S}(X_2, E_2)$ be the collections of all sets of X_1 and X_2, respectively. Suppose that $F_A \in \mathscr{S}(X_1, E_1)$ and $G_B \in \mathscr{S}(X_2, E_2)$ such that $A \subseteq E_1$ and $B \subseteq E_2$. If $\Phi_U : X_1 \to X_2$ and $\Phi_P : E_1 \to E_2$ are two mappings, where U, P denoted the set of the universe and the set of parameter respectively, then*

1. *The map $\Phi : \mathscr{S}(X_1, E_1) \to \mathscr{S}(X_2, E_2)$ is called a soft map from X_1 to X_2 for the soft set F_A, the soft set $\Phi(F_A) \in \mathscr{S}(X_2, E_2)$ is given by:*

$$(\Phi(F))(\varepsilon) = \begin{cases} \Phi_U(\cup_{e \in \Phi_P^{-1}(\varepsilon) \cap A} F(e)) & \text{if } \Phi_P^{-1}(\varepsilon) \cap A \neq \emptyset \\ \emptyset & \text{otherwise,} \end{cases} \tag{1.1.1}$$

for all $\varepsilon \in E_2$.

2. *The inverse image of the soft set G_B under soft mapping Φ is a soft set $\Phi^{-1}(G_B) \in \mathscr{S}(X_1, E_1)$ is defined by:*

$$(\Phi^{-1}(G))(e) = \begin{cases} \Phi_U^{-1}(G(\Phi_P(e))) & \text{if } \Phi_P(e) \in B \\ \emptyset & \text{otherwise}, \end{cases} \tag{1.1.2}$$

for any $e \in A \subseteq E_1$.

Definition 1.11 *(Kandil et al., 2011) An interval-valued fuzzy topological space (X, τ) is called:*

1 *An interval-valued fuzzy quasi T_0 space ($IVFq$-T_0 space) if every two IVF-points $x \neq y$ in X, there exists IVF-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

2 *An interval-valued fuzzy quasi T_1 space ($IVFq$-T_1 space) if every two IVF-points $x \neq y$ in X, there exists two IVF-open sets U and V such that $x \in U$ and $y \notin U$ and $y \in V$ and $x \notin V$.*

3 *An interval-valued fuzzy quasi T_2 space ($IVFq$-T_2 space) if every two IVF-points $x \neq y$ in X, there exists two disjoint IVF-open sets U and V such that $x \in U$ and $y \in V$, $U \tilde{\wedge} V = \emptyset$.*

4 *An interval-valued fuzzy quasi regular space ($IVFq$-regular space) if every IVF-closed set F and $x \in X$ such that $x \notin F$, there exists two disjoint IVF-open sets U and V such that $F \tilde{\subseteq} U$ and $x \in V$.*

5 *An interval-valued fuzzy quasi normal space ($IVFq$-normal space) if every two disjoint IVF-closed sets F and H, there exists two disjoint IVF-open sets U and V such that $F \tilde{\subseteq} U$ and $H \tilde{\subseteq} V$.*

6 *An interval-valued fuzzy quasi T_3 space ($IVFq$-T_3 space) if it is both $IVFq$-regular space and $IVFq$-T_1 space.*

7 *An interval-valued fuzzy quasi T_4 space ($IVFq$-T_4 space) if it is both IVF-normal space and $IVFq$-T_1 space.*

Using the matrix form of interval-valued fuzzy relations, authors in Rajarajeswari and Dhanalakshmi (2014) represented a finite IVFSs f_E as the following

Definition 1.12 *(Rajarajeswari and Dhanalakshmi, 2014) Let f_E be an $IVFS$ over X. The matrix*

$$f_E = \left[[f_{ij}^-, f_{ij}^+] \right]_{n \times m} = \begin{bmatrix} [f_{e_1}^-(x_1), f_{e_1}^+(x_1)] & \dots & [f_{e_1}^-(x_m), f_{e_1}^+(x_m)] \\ \vdots & \dots & \vdots \\ [f_{en}^-(x_1), f_{en}^+(x_1)] & \dots & [f_{en}^-(x_m), f_{en}^+(x_m)] \end{bmatrix}_{n \times m}$$

is called an $n \times m$ IVFS-matrix of the interval-valued fuzzy soft set f_E.

Here, $\mid E \mid = n$, $\mid X \mid = m$ and $f_{ij}^{-} = f_{e_i}^{-}(x_j)$, $f_{ij}^{+} = f_{e_i}^{+}(x_j)$ for $i = 1,...,n$ and $j = 1,...,m$.

Accordingly, the concepts of union, intersection, complement, etc., can be represented by matrix format in the finite case.

Proposition 1.2 *(Zulqarnain and Saeed, 2017) Let $[[f_{ij}^{-}, f_{ij}^{+}]]_{n\times m}$ be an interval-valued fuzzy matrix of interval-valued fuzzy soft set f_E. Then*

1. $[[0,0]]_{n\times m}^{c} = [[1,1]]_{n\times m}$.
2. $[[f_{ij}^{-}, f_{ij}^{+}]^{c}]_{n\times m}^{c} = [[f_{ij}^{-}, f_{ij}^{+}]]_{n\times m}$.
3. $([[f_{ij}^{-}, f_{ij}^{+}]]_{n\times m} \tilde{\cup} [[g_{ij}^{-}, g_{ij}^{+}]]_{n\times m})^{c} = [[f_{ij}^{-}, f_{ij}^{+}]]_{n\times m}^{c} \tilde{\cap} [[g_{ij}^{-}, g_{ij}^{+}]]_{n\times m}^{c}$.
4. $([[f_{ij}^{-}, f_{ij}^{+}]]_{n\times m} \tilde{\cap} [[g_{ij}^{-}, g_{ij}^{+}]]_{n\times m})^{c} = [[f_{ij}^{-}, f_{ij}^{+}]]_{n\times m}^{c} \tilde{\cup} [[g_{ij}^{-}, g_{ij}^{+}]]_{n\times m}^{c}$.

1.2 Problem Statement

The application of interval-valued fuzzy soft sets has been recently studied. Some research have been tried to adopt soft sets and interval-valued fuzzy topology. However, these methods cannot overcome the issue of comparable alternatives and might be ignored due to the lack of a comprehensive priority approach. As a result, these techniques have limited ability to evaluate options based on the preferences of decision-makers. To put it another way, while a linear ordering system is optimal in any group decision-making (GDM) problem, there are some real-world situations where some things are incomparable. In order to provide a partial solution to this problem, a non-linear ordered structure, such as a preorder relation or a preference relationship, may be utilized.

1.3 Objectives

The thesis's major goal is to provide a more efficient decision-making system based on interval-valued fuzzy soft sets. This research focuses on the development of interval-valued fuzzy soft theory and the application of interval-valued fuzzy soft sets to decision-making. To achieve this goal, we present the following objectives:

- To study interval-valued fuzzy soft topology.
- To introduce two different crisp sets called α-upper set and β-lower set of each parameter e and then, to construct two different crisp topologies called

α-upper-e topology, β-lower-e topology and investigate some properties of these two different crisp topological spaces.

- To investigate some binary relationships on the universal set X based on two different crisp topological spaces.
- To provide an algorithm for solving decision-making problems based on interval-valued fuzzy soft topology.

Figure 1.1 shows the contributions of this thesis.

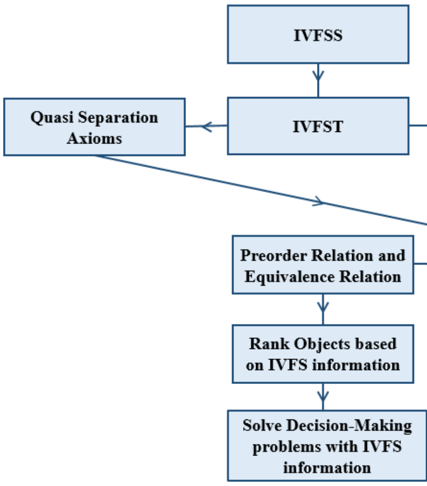

Figure 1.1: The diagram of contributions

1.4 Thesis Organization

The major goal of this thesis is to provide an interval-valued fuzzy soft topological approach for solving decision-making problems. The thesis is organized in the following way to attain this goal:

In Chapter 1, we give a broad overview of the numerous mathematical theories that deal with uncertainty. In addition, we provide background information on topology binary relations and decision-making theory. In this chapter, we also discuss the problem statement and objectives of this thesis.

Chapter 2 is a literature review, where we are particularly interested in interval-valued fuzzy soft sets and their use in decision-making.

Chapter 3 covers some theoretical aspects of our research on an interval-valued soft sets. Firstly, we present topological properties of interval-valued fuzzy soft sets, for instance, interval-valued fuzzy soft interior, interval-valued fuzzy soft closure, and fuzzy soft continuity. In addition, interval-valued fuzzy soft neighborhood and soft quasi neighborhood of the interval-valued fuzzy soft point are also studied. Interval-valued fuzzy soft quasi-separation axioms are studied in this chapter.

Chapter 4 studies several topological structures associated with interval-valued fuzzy soft topology. Firstly, we introduce two crisp sets, known as the lower crisp set and the upper crisp set of the interval-valued fuzzy soft set. Then, we built two topological spaces known as lower topology and upper topology on the universal set X using the upper and lower crisp sets. The link between these two topologies and interval-valued fuzzy soft quasi-separation axioms is constructed.

Chapter 5 provides some binary relations over the universal set X and shows that these are related to the previous results presented in Chapter 4.

The application part of our research is covered in Chapter 6. This chapter starts with the results of the matrix representations in chapters 4 and 5 and then continues to a method for classifying and ranking objects. Some algorithms for solving a ranking problem in decision-making are designed based on interval-valued fuzzy soft sets. We also validated our results by giving some illustrative examples.

Finally, chapter 7 presents a conclusion and recommendations related to the future development of this work.

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