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Algebraic representation of Three Qubit Quantum Circuit Problems

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Abstract

The evolution of quantum states serves as good fundamental studies in understanding the quantum information systems which finally lead to the research on quantum computation. To carry out such a study, mathematical tools such as the Lie group and their associated Lie algebra is of great importance. In this study, the Lie algebra of $\mathfrak{su}(8)$ is represented in a tensor product operation between three Pauli matrices. This can be realized by constructing the generalized Gell-Mann matrices and comparing them to the Pauli bases. It is shown that there is a one-to-one correlation of the Gell-Mann matrices with the Pauli basis which resembled the change of coordinates. Together with the commutator relations and the frequency analysis of the structure constant via the algebra, the Lie bracket operation will be highlighted providing insight into relating quantum circuit model with Lie Algebra. These are particularly useful when dealing with three-qubit quantum circuit problems which involve quantum gates that is derived from the $SU(8)$ Lie group.

Keywords: Lie group, Lie algebra, Riemannian geometry, quantum computation.

1 Introduction

Studies of quantum mechanics have become a staple foundation to have a good grasp on the fundamentals of nature in order to harness those power for future technologies. Some of the promising applications are the implementation of Shor's algorithm for prime factorization problem [14], and implementation of Grover's search algorithm together with a genetic algorithm for determining the graph's planarity [7], [10]. One of the important studies to apply quantum mechanics into a physically realizable system is by using group theory. As an example, observables such as energy, E can only be obtained from the application of operators in $SO(n)$ group to the quantum states as it ensures the eigenvalues to be real, while the evolution of quantum systems needed to be governed by operators in $SU(n)$ group to preserve the unit probability of the systems. Group theory also have wide application in physics, chemistry, and material sciences due to groups preserving the symmetry properties of the system which leads to invariant property. Some of the simple objects such as a circle arise naturally from group theory in particularly $e^{i\theta} \in U(1)$ where θ traced how the path of the circles. This concept was eventually extended to Lie group by Sophus Lie in 1884 to encompass ideas of the application of group theory to geometrical objects. This enable Lie group to have both group and geometry properties, which is particularly useful as certain mathematics problems in group theory can be recast into geometry or vice versa for different insights [3, 6, 8, 9, 16].

It is well-known that the Lie algebra is one of the representations of its associated Lie group, which simplify the calculation and computation of the problems. Using the previous example of $e^{i\theta}$ in Lie group which represents the global symmetry of a circle, its Lie algebra is a real line representing the local infinitesimal symmetry, which helped to understand the solutions of the equations. Since Lie algebra is a linear object, it could be constructed by linearly combining its basis. In particular $\mathfrak{su}(n)$ Lie algebra is constructed by the corresponding generalized Gell-Mann matrices. However, the choice of any basis is just the preferences of the user. Thus it would be possible to construct (or represent) $\mathfrak{su}(n)$ in another basis such as Clifford + T basis, which is the augmented Clifford group with T gate. Clifford group is defined as a group of unitaries that normalize the Pauli group, and it is generated by Hadamard, S and CNOT gates. In our work instead, we work on the basis of the tensor product of Pauli matrices for $\mathfrak{su}(2^n)$, as it encompasses the full space of entanglement for the quantum circuit representation. The motivation behind this representation is due to 1) it is much easier to compute the tensor product of Pauli matrices as compared to generalized Gell-Mann matrices, 2) for certain problems (quantum circuit problems), having Pauli coordinates give better insights in the evolution of states contributed by quantum gates. This is particularly useful in representing the evolution of the quantum states, $U = e^{-iHt}$ using Riemannian geometry tools. This is done by representing the Hamiltonian, H of the evolution of the quantum states in terms of Pauli coordinates where U is an element of $SU(8)$ Lie group, and H is also the respective element for $\mathfrak{su}(8)$ Lie algebra [4, 5, 12].

In particular, it is explicitly shown that the decomposition of the n-qubit system with the tensor product of Pauli matrices. It is shown to be one-to-one correspondence or just a change of basis to the generalized Gell-Mann matrices. In line with motivation (2), we, therefore, prepare this work such that, the proper evolution of quantum states in terms of Pauli coordinates will be properly understood. To our knowledge, the Lie algebra of d-dimensional qudits system ($\mathfrak{su}(d)$) can be decomposed by the either three different matrix bases for example the generalized Gell-Mann matrix basis, the polarization operator basis and the Weyl operator basis [2]. From another perspective, an n-qubit system resembled a 2^n-dimensional qudits system will have its associated $\mathfrak{su}(2^n)$ Lie algebra which can also be decomposed by those matrix bases.

This paper has the following structure. Section 2 describes the general form for $SU(n)$ Lie group namely $SU(8)$ and provides the counting method for the number of free parameters for $SU(8)$. Next the generator for $SU(8)$ is constructed, denoted by $\mathfrak{su}(8)$ Lie algebra having a well-known form to be the Gell-Mann matrices. In Section 3, the tensor product of Pauli matrices is computed as the Pauli basis and they are categorized according to their forms. Relation between generators of $SU(8)$ with tensor product of Pauli matrices associated with their forms will also be given in this section. This is followed by highlighting the concept of change of basis and a quick computation method for respective basis coefficients. Section 4 will be covering some properties of $\mathfrak{su}(8)$ on Pauli basis as well as the comparisons with $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$ Lie algebra for 1-qubit and 2-qubit system. The final section shall offer a concise recap and conclusion.

2 Construction of the generator for $SU(8)$

2.1 General form for $\mathbf{SU(n)}$

To construct a general form for $SU(8)$ Lie group, any element of $SU(8)$, should satisfy these conditions:

1. The diagonals must be real.
2. The diagonals must be traceless.
3. The upper triangle of the matrix must be the complex conjugate of the lower triangle of the matrix.

These conditions are imposed by the Hermiticity of $SU(n)$. With that, the number of free parameters for $SU(n)$ can be calculated:

1. There is $2n^2$ free parameters due to $n \times n$ complex entries of $SU(n)$ matrix.
2. The n diagonals must be real thus removing n free parameters.
3. Due to condition 3., the upper triangle have $2(n^2-n)/2$ components, where n^2-n is obtained by removing the n diagonals from n^2 components of matrix, multiplied by 2 due to complex component and divided by 2 to remove the lower triangle component. With this condition we removed $2(n^2-n)/2$ components.
4. By applying step 1., 2. and 3. we obtained the number of free parameters for $SU(n)$ Lie group:
$$2n^2 - n - \frac{2(n^2-n)}{2} = n^2.$$
5. Finally due to "special" properties of $SU(8)$ there is an additional restrains which is $\det(SU(n)) = 1$ thus removing 1 more free parameters yielding n^2-1 free parameters.

Given our case are $SU(8)$ we have $8^2-1 = 63$ free parameters. One of the general form for $SU(8)$ Lie group, G is as follows:

$$G = \begin{pmatrix} e^{i\psi_1} & a_1 - ib_1 & \dots & a_6 - ib_6 & a_7 - ib_7 \\ a_1 + ib_1 & e^{i\psi_2} & \dots & a_{12} - ib_{12} & a_{13} - ib_{13} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_6 + ib_6 & a_{12} + ib_{12} & \dots & e^{i\psi_7} & a_{28} - ib_{28} \\ a_7 + ib_7 & a_{13} + ib_{13} & \dots & a_{28} + ib_{28} & e^{i\psi_8} \end{pmatrix}.$$

With the same "special" condition of $SU(8)$, $\det(G) = 1$ which lead to sum of the exponential part of the diagonal of G must be zero, with the following condition:

$$\det(G) = \det\left(e^{tX}\right) = e^{t\textbf{Tr}(X)},$$

where **Tr**(X) is the trace of matrix X, thus $\psi_8 = -\psi_1 - \psi_2 - \cdots - \psi_7$ is required. With that, we have ψ_1 to ψ_7, 7 parameter, a_1 to a_{28}, 28 parameter, b_1 to b_{28}, 28 parameter, lead to a total of 7+28+28=63 free parameters.

2.2 Generator construction

Let G be a matrix Lie group. Then the Lie algebra of G denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for all real numbers t [8].
Since:

$$\left.\frac{d}{dt}\right|_{t=0} e^{tX} = X,$$

we can obtained the generator, X. For the case of $SU(8)$, its generators is the element of $\mathfrak{su}(8)$ where $G \in SU(8)$ is generated by linear combination of those 63 free parameters thus having 63 generators, labeled as $X_i \in \mathfrak{su}(8)$, where $i = 1, 2, \ldots, 63$. Thus G can be written as:

$$G = \exp(\psi_1 X_1 + \cdots + \psi_7 X_7 + a_1 X_8 + \cdots + a_{28} X_{35} + b_1 X_{36} + \cdots + b_{28} X_{63}).$$

Following the above, the 63 generators can be found as such, for the diagonal part, we have:

$$\left.\frac{\partial G}{\partial \psi_1}\right|_{\psi_1=\cdots=\psi_7=a_1=\cdots=a_{28}=b_1=\cdots=b_{28}=0} = X_1,$$

$$\vdots$$

$$\left.\frac{\partial G}{\partial \psi_7}\right|_{\psi_1=\cdots=\psi_7=a_1=\cdots=a_{28}=b_1=\cdots=b_{28}=0} = X_7,$$

and for the off-diagonal real parts, the respective generators $X_8, \ldots, X_{35}$ can be obtained by taking partial derivative of respective parameters, $a_1, \ldots, a_{28}$. While for off-diagonal imaginary parts, the respective generators $X_{36}, \ldots, X_{63}$ can be obtained by taking partial derivative of respective parameters, $b_1, \ldots, b_{28}$. X_i, $i = 8, 9, \ldots, 63$ are the generalized Gell-Mann matrices for $SU(8)$, while X_i, $i = 1, 2, \ldots 7$ can be linearly combined to obtain the diagonal part for the generalized Gell-Mann matrices for $SU(8)$.

3 Representation of $\mathfrak{su}(8)$ in Pauli basis

3.1 Pauli basis

Let us start constructing Pauli basis for $\mathfrak{su}(8)$. Pauli Matrices can be categorized in two groups which are the diagonals (labeled as D):

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the off-diagonals (labeled as OD):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

Thus the tensor product of Pauli matrices can come in eight forms by permutation as follows:

$$\begin{aligned} D \otimes D \otimes D, &\quad D \otimes D \otimes OD, \\ D \otimes OD \otimes D, &\quad D \otimes OD \otimes OD, \\ OD \otimes D \otimes D, &\quad OD \otimes D \otimes OD, \\ OD \otimes OD \otimes D, &\quad OD \otimes OD \otimes OD. \end{aligned}$$

A permutation of 4 times for each qubits contribute to $4 \times 4 \times 4 = 64$ forms of Pauli matrices. Then minus one out due to $\det(SU(n)) = 1$ resulting 63 tensor product of Pauli matrices. These form the Pauli basis for $SU(8)$. As an example, $D \otimes D \otimes OD$:

$$\begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \end{pmatrix},$$

where $\alpha \in \{1, i, -1, -i\}$ being the quaternion. And one of the element from $D \otimes D \otimes OD$ form is as followed:

$$\sigma_z \otimes I \otimes \sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix}.$$

The full list of $\mathfrak{su}(8)$ in Pauli basis are as follows:

$$\begin{aligned} S(H) = & a_1 I \otimes I \otimes \sigma_x + a_2 I \otimes I \otimes \sigma_y + a_3 I \otimes I \otimes \sigma_z \\ & + a_4 I \otimes \sigma_x \otimes I + a_5 I \otimes \sigma_y \otimes I + a_6 I \otimes \sigma_z \otimes I \\ & + a_7 \sigma_x \otimes I \otimes I + a_8 \sigma_y \otimes I \otimes I + a_9 \sigma_z \otimes I \otimes I \end{aligned}$$

$$\begin{aligned} T(H) = & a_{10} I \otimes \sigma_x \otimes \sigma_x + a_{11} I \otimes \sigma_x \otimes \sigma_y + a_{12} I \otimes \sigma_x \otimes \sigma_z \\ & + a_{13} I \otimes \sigma_y \otimes \sigma_x + a_{14} I \otimes \sigma_y \otimes \sigma_y + a_{15} I \otimes \sigma_y \otimes \sigma_z \\ & + a_{16} I \otimes \sigma_z \otimes \sigma_x + a_{17} I \otimes \sigma_z \otimes \sigma_y + a_{18} I \otimes \sigma_z \otimes \sigma_z \\ & + a_{19} \sigma_x \otimes I \otimes \sigma_x + a_{20} \sigma_x \otimes I \otimes \sigma_y + a_{21} \sigma_x \otimes I \otimes \sigma_z \\ & + a_{22} \sigma_y \otimes I \otimes \sigma_x + a_{23} \sigma_y \otimes I \otimes \sigma_y + a_{24} \sigma_y \otimes I \otimes \sigma_z \\ & + a_{25} \sigma_z \otimes I \otimes \sigma_x + a_{26} \sigma_z \otimes I \otimes \sigma_y + a_{27} \sigma_z \otimes I \otimes \sigma_z \\ & + a_{28} \sigma_x \otimes \sigma_x \otimes I + a_{29} \sigma_x \otimes \sigma_y \otimes I + a_{30} \sigma_x \otimes \sigma_z \otimes I \\ & + a_{31} \sigma_y \otimes \sigma_x \otimes I + a_{32} \sigma_y \otimes \sigma_y \otimes I + a_{33} \sigma_y \otimes \sigma_z \otimes I \\ & + a_{34} \sigma_z \otimes \sigma_x \otimes I + a_{35} \sigma_z \otimes \sigma_y \otimes I + a_{36} \sigma_z \otimes \sigma_z \otimes I \end{aligned}$$

$$\begin{aligned}Q(H) =& a_{37}\sigma_x \otimes \sigma_x \otimes \sigma_x + a_{38}\sigma_x \otimes \sigma_x \otimes \sigma_y + a_{39}\sigma_x \otimes \sigma_x \otimes \sigma_z \\ &+ a_{40}\sigma_x \otimes \sigma_y \otimes \sigma_x + a_{41}\sigma_x \otimes \sigma_y \otimes \sigma_y + a_{42}\sigma_x \otimes \sigma_y \otimes \sigma_z \\ &+ a_{43}\sigma_x \otimes \sigma_z \otimes \sigma_x + a_{44}\sigma_x \otimes \sigma_z \otimes \sigma_y + a_{45}\sigma_x \otimes \sigma_z \otimes \sigma_z \\ &+ a_{46}\sigma_y \otimes \sigma_x \otimes \sigma_x + a_{47}\sigma_y \otimes \sigma_x \otimes \sigma_y + a_{48}\sigma_y \otimes \sigma_x \otimes \sigma_z \\ &+ a_{49}\sigma_y \otimes \sigma_y \otimes \sigma_x + a_{50}\sigma_y \otimes \sigma_y \otimes \sigma_y + a_{51}\sigma_y \otimes \sigma_y \otimes \sigma_z \\ &+ a_{52}\sigma_y \otimes \sigma_z \otimes \sigma_x + a_{53}\sigma_y \otimes \sigma_z \otimes \sigma_y + a_{54}\sigma_y \otimes \sigma_z \otimes \sigma_z \\ &+ a_{55}\sigma_z \otimes \sigma_x \otimes \sigma_x + a_{56}\sigma_z \otimes \sigma_x \otimes \sigma_y + a_{57}\sigma_z \otimes \sigma_x \otimes \sigma_z \\ &+ a_{58}\sigma_z \otimes \sigma_y \otimes \sigma_x + a_{59}\sigma_z \otimes \sigma_y \otimes \sigma_y + a_{60}\sigma_z \otimes \sigma_y \otimes \sigma_z \\ &+ a_{61}\sigma_z \otimes \sigma_z \otimes \sigma_x + a_{62}\sigma_z \otimes \sigma_z \otimes \sigma_y + a_{63}\sigma_z \otimes \sigma_z \otimes \sigma_z\end{aligned}$$

where $S(H)$ consist of one, $T(H)$ consist of two, and $Q(H)$ consist of three Pauli matrices tensor product respectively, which have weight of 1, 2, and 3 respectively. The number of generalize Pauli matrices with weight k computed agreed with the following formula by [1]:

$$\mathcal{N}_k = 3^k \binom{n}{k},$$

where n is the number of qubits of the system.

3.2 Representing $\mathfrak{su}(8)$ in Pauli basis

The generator can be represented in the categories of the form in the previous section. The results are as follows:

Table 1: Categorization of generator of $\mathfrak{su}(8)$ in Pauli basis forms

Generator	Form			
	$D \otimes D \otimes D$	$D \otimes D \otimes OD$	$D \otimes OD \otimes D$	$D \otimes OD \otimes OD$
Real	$X_1, X_2, X_3, X_4,$ X_5, X_6, X_7	X_8, X_{21} X_{30}, X_{35}	X_9, X_{16} X_{31}, X_{34}	X_{10}, X_{15} X_{32}, X_{33}
Imaginary		X_{36}, X_{49} X_{58}, X_{63}	X_{37}, X_{44} X_{59}, X_{62}	X_{38}, X_{43} X_{60}, X_{61}

Generator	Form			
	$OD \otimes D \otimes D$	$OD \otimes D \otimes OD$	$OD \otimes OD \otimes D$	$OD \otimes OD \otimes OD$
Real	X_{11}, X_{18} X_{24}, X_{29}	X_{12}, X_{17} X_{25}, X_{28}	X_{13}, X_{20} X_{22}, X_{27}	X_{14}, X_{19} X_{23}, X_{26}
Imaginary	X_{39}, X_{46} X_{52}, X_{57}	X_{40}, X_{45} X_{53}, X_{56}	X_{41}, X_{48} X_{50}, X_{55}	X_{42}, X_{47} X_{51}, X_{54}

As an example:

$$\begin{aligned}X_1 = \frac{1}{4}\big[&(I \otimes I \otimes \sigma_z) + (\sigma_z \otimes I \otimes I) \\ &+ (I \otimes \sigma_z \otimes I) + (\sigma_z \otimes \sigma_z \otimes \sigma_z)\big].\end{aligned}$$

3.3 Change of Basis from Generator Basis to Pauli Basis

The result in Table 1 shown to have one-to-one correspondence between generator of $SU(8)$ Lie group and tensor product of Pauli matrices. According to change of basis equation:

$$y^\mu = g^\mu_\nu x^\nu,$$

where $y^\mu, x^\nu, \mu, \nu = 1, 2, 3, 4$ are different Lie algebra basis, g^μ_ν is the transformation matrix, Given the generator of $SU(8)$, the Hamiltonian, H bases transform orthogonally, one can get away without worrying about the distinction between covariant and contravariant part of the tensor, and also it satisfied the properties of $(g^\mu_\nu)^{-1} = (g^\mu_\nu)^T = g^\nu_\mu$. Let us look at this example:
$D \otimes D \otimes OD$ form generator:
Real part :

$$\begin{aligned}
I \otimes I \otimes \sigma_x &= X_8 + X_{21} + X_{30} + X_{35},\\
I \otimes \sigma_z \otimes \sigma_x &= X_8 - X_{21} + X_{30} - X_{35},\\
\sigma_z \otimes I \otimes \sigma_x &= X_8 + X_{21} - X_{30} - X_{35},\\
\sigma_z \otimes \sigma_z \otimes \sigma_x &= X_8 - X_{21} - X_{30} + X_{35}.
\end{aligned}$$

Let:

$$\begin{aligned}
y^1 &= I \otimes I \otimes \sigma_x, & x^1 &= X_8,\\
y^2 &= I \otimes \sigma_z \otimes \sigma_x, & x^2 &= X_{21},\\
y^3 &= \sigma_z \otimes I \otimes \sigma_x, & x^3 &= X_{30},\\
y^4 &= \sigma_z \otimes \sigma_z \otimes \sigma_x, & x_4 &= X_{35},
\end{aligned}$$

one can rewrite the equations:

$$\begin{aligned}
y^\mu &= g^\mu_\nu x^\nu,\\
\begin{pmatrix} y^1\\ y^2\\ y^3\\ y^4 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x^1\\ x^2\\ x^3\\ x^4 \end{pmatrix}.
\end{aligned}$$

By finding the inverse of g^μ_ν one can change the basis of x^ν which is the generator of $SU(8)$ back to tensor product of Pauli matrices.

$$\begin{aligned}
(g^\mu_\nu)^{-1} y^\mu &= (g^\mu_\nu)^{-1} g^\mu_\nu x^\nu,\\
x^\nu &= (g^\mu_\nu)^{-1} y^\mu,\\
x^\nu &= g^\nu_\mu y^\mu,\\
\begin{pmatrix} x^1\\ x^2\\ x^3\\ x^4 \end{pmatrix} &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y^1\\ y^2\\ y^3\\ y^4 \end{pmatrix},
\end{aligned}$$

which is as followed

$$X_8 = \frac{1}{4}\Big[(I \otimes I \otimes \sigma_x) + (I \otimes \sigma_z \otimes \sigma_x) + (\sigma_z \otimes I \otimes \sigma_x) + (\sigma_z \otimes \sigma_z \otimes \sigma_x)\Big],$$

$$X_{21} = \frac{1}{4}\Big[(I \otimes I \otimes \sigma_x) - (I \otimes \sigma_z \otimes \sigma_x) + (\sigma_z \otimes I \otimes \sigma_x) - (\sigma_z \otimes \sigma_z \otimes \sigma_x)\Big],$$

$$X_{30} = \frac{1}{4}\Big[(I \otimes I \otimes \sigma_x) + (I \otimes \sigma_z \otimes \sigma_x) - (\sigma_z \otimes I \otimes \sigma_x) - (\sigma_z \otimes \sigma_z \otimes \sigma_x)\Big],$$

$$X_{35} = \frac{1}{4}\Big[(I \otimes I \otimes \sigma_x) - (I \otimes \sigma_z \otimes \sigma_x) - (\sigma_z \otimes I \otimes \sigma_x) + (\sigma_z \otimes \sigma_z \otimes \sigma_x)\Big].$$

Note that $(g^{\mu}_{\nu})^{-1} = g^{\nu}_{\mu}$, when:

$$g^{\mu}_{\nu} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = g^{\nu}_{\mu},$$

which is different than the one before by a factor of $\frac{1}{2}$. This is still acceptable as the basis are normalized unit, thus changing the magnitude would not affects the properties of a basis.

From the result, it is shown that all 63 generators of $SU(8)$ Lie group can be represented in the linear combination of the tensor product of Pauli matrices. All 63 generators have unique linear combination which is one-to-one correspondence similar to change of basis. Furthermore, each set of four generators is generated by a set of four tensor product of Pauli matrices by linear combination. This is due to fixing one of the Pauli matrices while permuting the other two Pauli matrices will produce four different tensor products with the same form. As for the factor of differentiating real and imaginary generator come down to the contribution of off-diagonal part, σ_x and σ_y. This further implies the similarity of change of basis. In addition to this, there was no contribution of $I \otimes I \otimes I$ component in any linear combination due to traceless properties of the generator of $SU(8)$. The results are especially important, as it shown that we could represent the Lie algebra $\mathfrak{su}(8)$ one-to-one in Pauli coordinates resembling change of coordinates which greatly reduced the amount of terms used for representation [12, 5, 4].

3.4 Computation of Coefficient for Respective Pauli Basis

Now that we can represent the generator of $SU(8)$ in Pauli basis, we can find the coefficient for the respective basis with a mere simple method. The Hamiltonian of the quantum circuit, H is the generator of $SU(8)$, and σ_i, is the tensor product of Pauli matrices where $i = 1, 2, \ldots, 63$ represent the basis, we can compute the coefficient for the respective Pauli basis as follows:

$$\mathrm{tr}(\sigma_i.H)\begin{cases} \neq 0 & \text{if } \sigma_i \text{ is its basis} \\ = 0 & \text{if } \sigma_i \text{ is not its basis} \end{cases}$$

4 Properties of $\mathfrak{su}(8)$ in the Pauli basis

The basis of $\mathfrak{su}(8)$, are all orthogonal and satisfying:

$$\mathrm{Tr}\left(\sigma_i^2\right) = 8.$$

Elements of $\mathfrak{su}(8)$ represented in Pauli basis obey the following commutation relation

$$\begin{aligned}[\sigma_a \otimes \sigma_d \otimes \sigma_g, \sigma_b \otimes \sigma_e \otimes \sigma_f] =& 2[(i\epsilon_{abc})(i\epsilon_{def})(i\epsilon_{ghi})(\sigma_d \otimes \sigma_f \otimes \sigma_i)] \\ &+ \delta_{ab}\delta_{cd}(I \otimes I \otimes i\epsilon_{ghi}\sigma_i) \\ &+ \delta_{ab}\delta_{gh}(I \otimes i\epsilon_{def}\sigma_f \otimes I) \\ &+ \delta_{cd}\delta_{gh}(i\epsilon_{abc}\sigma_c \otimes I \otimes I),\end{aligned}$$

where, $\sigma_a, \sigma_b, \sigma_c, \sigma_d, \sigma_e, \sigma_f, \sigma_g, \sigma_h, \sigma_i$ are element of Pauli matrices, $\delta_{ab}, \delta_{cd}, \delta_{gh}$ are Kronecker Delta, and $\epsilon_{abc}, \epsilon_{def}, \epsilon_{ghi}$ are Levi-Civita Symbol. In a more condensed form, we can write as follows:

$$[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{63} c_{ij}^k \sigma_k,$$

with $i, j, k = 1, \ldots, 63$ and c_{ij}^k is the respective structure constants. It also fulfill the Jacobi identity:

$$[[\sigma_i, \sigma_j], \sigma_k] + [[\sigma_j, \sigma_k], \sigma_i] + [[\sigma_k, \sigma_i], \sigma_j] = 0,$$

and

$$\mathrm{Tr}(\sigma_i[\sigma_j, \sigma_k]) = 16ic_{ij}^k.$$

These computation results are in conjunction with the $SU(4)$ Lie group studies done by [13].

For $\mathfrak{su}(8)$ there exist 63 generators which in this work are represented in the tensor product of Pauli matrices, thus there are 63×63 commutation relations contributing to 3969 possible structure constants. Structure constants can come with three possible values which are 1, -1, and 0, and the frequency of respective structure constants value are as follows:

Let S_{11} as the notation be defined as square section for row 1, column 1 of commutation relation table from $\sigma_1, \ldots, \sigma_{63}$:
The full computation of the structure constants for $\mathfrak{su}(8)$ can be found in the Appendix. Structure constants for $\mathfrak{su}(8)$ are anti-symmetric in nature. From Table 2, it is clearly shown that the one-qubit with three-qubits and two-qubits with two-qubits commutation relations has the highest frequency of non-commutating Lie bracket operation. This has a correlation with quantum physics operator theory where useful evolutions or measurements on states require non-commutative operators as it incites entanglement between qubits. Thus it shows the importance of choices of applications of operators (gates) in a quantum circuit.

Next, let us compare with the lower rank of Lie algebra $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$ for one qubit and two-qubit quantum circuitry:

From the comparison of Table 3, it is clearly shown that as the number of qubits in quantum circuitry increases, the structure constants space increasingly become larger, scaling with $(2^n - 1)^2$. The percentage of non-zero structure constants over the entire structure constants space dropped, which indicated non-entangling gates of choices increase as the number of qubits increase.
The entire list of structure constants can be referred from Appendix A. Note that since the structure constants themselves are anti-symmetric, thus the table in Appendix A only show one half of the

Table 2: Breakdown for the frequency of $\mathfrak{su}(8)$ structure constants

One-qubit with one-qubit:

Frequency of c_{ij}^k value	1	-1	0
S_{11}	9	9	63

One-qubit with two-qubits:

Frequency of c_{ij}^k value	1	-1	0
S_{12}, S_{13}, S_{14} S_{21}, S_{31}, S_{41}	18	18	45

One-qubit with three-qubits:

Frequency of c_{ij}^k value	1	-1	0
S_{15}, S_{16}, S_{17} S_{51}, S_{61}, S_{71}	27	27	27

Two-qubits with two-qubits:

Frequency of c_{ij}^k value	1	-1	0
S_{22}, S_{33}, S_{44}	18	18	45
S_{23}, S_{24}, S_{34} S_{32}, S_{42}, S_{43}	27	27	27

Two-qubits with three-qubits:

Frequency of c_{ij}^k value	1	-1	0
S_{25}, S_{26}, S_{27} S_{52}, S_{62}, S_{72}	18	18	45
S_{35}, S_{36}, S_{37} S_{53}, S_{63}, S_{73}	18	18	45
S_{45}, S_{46}, S_{47} S_{54}, S_{64}, S_{74}	18	18	45

Three-qubits with three-qubits:

Frequency of c_{ij}^k value	1	-1	0
S_{55}, S_{66}, S_{77}	18	18	45
S_{56}, S_{67} S_{75}	27	18	36
S_{57} S_{65}, S_{76}	18	27	36

Table 3: Frequency of structure constants value

$\mathfrak{su}(2)$	c_{ij}^k value		
	1	-1	0
Frequency	3	3	3
% over 9	33.3%	33.3%	33.3%

$\mathfrak{su}(4)$	c_{ij}^k value		
	1	-1	0
Frequency	60	60	105
% over 225	26.7%	26.7%	46.7%

$\mathfrak{su}(8)$	c_{ij}^k value		
	1	-1	0
Frequency	1008	1008	1953
% over 3969	25.4%	25.4%	49.2%

structure constants without the anti-symmetric counter part. The computation results agreed with [1], due to the form for the weight of the commutator $[\sigma_i, \sigma_j]$:

$$w_r = |M - N| + 1 + 2r,$$

where M and N are the weight of σ_i and σ_j respectively, such that

$$r = \begin{cases} 0, \ldots, M-1 & M \geq N \\ 0, \ldots, N-1 & M < N \end{cases}.$$

From the formula and structure constants computed, we can see that the commutator between two weight 1 basis will only yield a weight 1 basis, while the commutator between two weight 3 basis can only obtain weight 1 or 3 basis. The only way to obtain the weight 2 basis is by taking the commutator between weight 1 and weight 3 basis, or between weight 2 and weight 2 basis. The existence of weight 2 basis enable to generate all weight 1, 2 and 3 basis state, thus showing the weight 2 state in $\mathfrak{su}(8)$. This is correlated to [11] that all quantum circuits can be essentially generated using one-qubit gates and CNOT gates.

5 Conclusion

In summary, the generators for $SU(8)$ Lie group which is $\mathfrak{su}(8)$ Lie algebra is computed as the generalized Gell-Mann matrices. We also have computed the tensor product of three Pauli matrices and categorized them. We then compared both of them and computed the representation from one to another. The results show to be a change of basis between generalized Gell-Mann matrices and the tensor product of three Pauli matrices. The structure constants for $\mathfrak{su}(8)$ are computed and compared with $\mathfrak{su}(4)$ and $\mathfrak{su}(2)$, and it shows that more qubits resulted in more non-entangling gate operations. Thus $\mathfrak{su}(8)$ can be represented in Pauli coordinates. From the structure constants of $\mathfrak{su}(8)$ we can clearly see the importance of the two-qubits gate in a quantum circuit. Through our frequency analysis for the structure constant for $\mathfrak{su}(8)$, one can further investigate it through the tunnel of Cartan subalgebra where it could further analyse the abelian subgroup and the non-abelian subgroup for the suggestion of gates for quantum circuit problems. On the same page, one can consider another approach of using Lie symmetry analysis [15] to obtain an exact solution to the differential equations, further enhancing the analysis for Lie Algebra.

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A Appendix

Structure Constant $c^k_{i(j)}$ for $\mathfrak{su}(8)$

One qubit commutation relation:

$c^k_{i(j)}$ value	1	-1	0
S_{11}	$c^3_{1(2)}, c^1_{2(3)}, c^2_{3(1)}$ $c^6_{4(5)}, c^4_{5(6)}, c^5_{6(4)}$ $c^9_{7(8)}, c^7_{8(9)}, c^8_{9(7)}$	$c^3_{2(1)}, c^1_{3(2)}, c^2_{1(3)}$ $c^6_{5(4)}, c^4_{6(5)}, c^5_{4(6)}$ $c^9_{8(7)}, c^7_{9(8)}, c^8_{7(9)}$	others

One qubit with two qubit commutation relation:

$c^k_{i(j)}$ value	1	-1	0
S_{12}	$c^{12}_{1(11)}, c^{15}_{1(14)}, c^{18}_{1(17)}$ $c^{10}_{2(12)}, c^{13}_{2(15)}, c^{16}_{2(18)}$ $c^{11}_{3(10)}, c^{14}_{3(13)}, c^{17}_{3(16)}$ $c^{16}_{4(13)}, c^{17}_{4(14)}, c^{18}_{4(15)}$ $c^{10}_{5(16)}, c^{11}_{5(17)}, c^{12}_{5(18)}$ $c^{13}_{6(10)}, c^{14}_{6(11)}, c^{15}_{6(12)}$	$c^{11}_{1(12)}, c^{14}_{1(15)}, c^{17}_{1(18)}$ $c^{12}_{2(10)}, c^{15}_{2(13)}, c^{18}_{2(16)}$ $c^{10}_{3(11)}, c^{13}_{3(14)}, c^{16}_{3(17)}$ $c^{13}_{4(16)}, c^{14}_{4(17)}, c^{15}_{4(18)}$ $c^{16}_{5(10)}, c^{17}_{5(11)}, c^{18}_{5(12)}$ $c^{10}_{6(13)}, c^{11}_{6(14)}, c^{12}_{6(15)}$	others
S_{13}	$c^{21}_{1(20)}, c^{24}_{1(23)}, c^{27}_{1(26)}$ $c^{19}_{2(21)}, c^{22}_{2(24)}, c^{25}_{2(27)}$ $c^{20}_{3(19)}, c^{23}_{3(22)}, c^{26}_{3(25)}$ $c^{25}_{7(22)}, c^{26}_{7(23)}, c^{27}_{7(24)}$ $c^{19}_{8(25)}, c^{20}_{8(26)}, c^{21}_{8(27)}$ $c^{22}_{9(19)}, c^{20}_{9(23)}, c^{21}_{9(24)}$	$c^{21}_{1(20)}, c^{23}_{1(24)}, c^{26}_{1(27)}$ $c^{21}_{2(19)}, c^{24}_{2(22)}, c^{27}_{2(25)}$ $c^{19}_{3(20)}, c^{22}_{3(23)}, c^{25}_{3(26)}$ $c^{22}_{7(25)}, c^{23}_{7(26)}, c^{24}_{7(27)}$ $c^{25}_{8(19)}, c^{26}_{8(20)}, c^{27}_{8(21)}$ $c^{19}_{9(22)}, c^{23}_{9(20)}, c^{24}_{9(21)}$	others
S_{14}	$c^{30}_{4(29)}, c^{33}_{4(32)}, c^{36}_{4(35)}$ $c^{28}_{5(30)}, c^{31}_{5(33)}, c^{34}_{5(36)}$ $c^{29}_{6(28)}, c^{32}_{6(31)}, c^{35}_{6(34)}$ $c^{34}_{7(31)}, c^{35}_{7(32)}, c^{36}_{7(33)}$ $c^{28}_{8(34)}, c^{29}_{8(35)}, c^{30}_{8(36)}$ $c^{31}_{9(28)}, c^{32}_{9(29)}, c^{33}_{9(30)}$	$c^{29}_{4(30)}, c^{32}_{4(33)}, c^{35}_{1(36)}$ $c^{30}_{5(28)}, c^{33}_{5(31)}, c^{36}_{5(34)}$ $c^{28}_{6(29)}, c^{31}_{6(32)}, c^{34}_{6(35)}$ $c^{31}_{7(34)}, c^{32}_{7(35)}, c^{33}_{7(36)}$ $c^{34}_{8(28)}, c^{35}_{8(29)}, c^{36}_{8(30)}$ $c^{28}_{9(31)}, c^{29}_{9(32)}, c^{30}_{9(33)}$	others

Malaysian J. Math. Sci. 16(3): 559–582 (2022) 559 - 582

One qubit with three qubit commutation relation:

$c^{k}_{i(j)}$ value	1	-1	0
S_{15}	$c^{39}_{1(38)}, c^{42}_{1(41)}, c^{45}_{1(44)}$	$c^{38}_{1(39)}, c^{41}_{1(42)}, c^{44}_{1(45)}$	
	$c^{37}_{2(39)}, c^{40}_{2(42)}, c^{43}_{2(45)}$	$c^{39}_{2(37)}, c^{42}_{2(40)}, c^{45}_{2(43)}$	
	$c^{38}_{3(37)}, c^{41}_{3(40)}, c^{44}_{3(43)}$	$c^{37}_{3(38)}, c^{40}_{3(41)}, c^{43}_{3(44)}$	
	$c^{43}_{4(40)}, c^{44}_{4(41)}, c^{45}_{4(42)}$	$c^{43}_{4(40)}, c^{41}_{4(44)}, c^{42}_{4(45)}$	
	$c^{37}_{5(43)}, c^{38}_{5(44)}, c^{39}_{5(45)}$	$c^{43}_{5(37)}, c^{44}_{5(38)}, c^{45}_{5(39)}$	others
	$c^{40}_{6(37)}, c^{41}_{6(38)}, c^{42}_{6(39)}$	$c^{37}_{6(40)}, c^{38}_{6(41)}, c^{39}_{6(42)}$	
	$c^{46}_{9(37)}, c^{47}_{9(38)}, c^{48}_{9(39)}$	$c^{55}_{8(37)}, c^{56}_{8(38)}, c^{57}_{8(39)}$	
	$c^{49}_{9(40)}, c^{50}_{9(41)}, c^{51}_{9(42)}$	$c^{58}_{8(40)}, c^{59}_{8(41)}, c^{60}_{8(42)}$	
	$c^{52}_{9(43)}, c^{53}_{9(44)}, c^{54}_{9(45)}$	$c^{61}_{8(43)}, c^{62}_{8(44)}, c^{63}_{8(45)}$	
S_{16}	$c^{48}_{1(47)}, c^{51}_{1(50)}, c^{54}_{1(53)}$	$c^{47}_{1(48)}, c^{50}_{1(51)}, c^{53}_{1(54)}$	
	$c^{46}_{2(48)}, c^{49}_{2(51)}, c^{52}_{2(54)}$	$c^{48}_{2(46)}, c^{51}_{2(49)}, c^{52}_{2(54)}$	
	$c^{47}_{3(46)}, c^{50}_{3(49)}, c^{53}_{3(52)}$	$c^{46}_{3(47)}, c^{49}_{3(50)}, c^{53}_{3(52)}$	
	$c^{52}_{4(49)}, c^{53}_{4(50)}, c^{54}_{4(51)}$	$c^{49}_{4(52)}, c^{50}_{4(53)}, c^{51}_{4(54)}$	
	$c^{46}_{5(52)}, c^{47}_{5(53)}, c^{48}_{5(54)}$	$c^{52}_{5(46)}, c^{53}_{5(47)}, c^{54}_{5(48)}$	others
	$c^{49}_{6(46)}, c^{50}_{6(47)}, c^{51}_{6(48)}$	$c^{46}_{6(49)}, c^{47}_{6(50)}, c^{48}_{6(51)}$	
	$c^{55}_{7(46)}, c^{56}_{7(47)}, c^{57}_{7(48)}$	$c^{37}_{9(46)}, c^{38}_{9(47)}, c^{39}_{9(48)}$	
	$c^{58}_{7(49)}, c^{59}_{7(50)}, c^{60}_{7(51)}$	$c^{40}_{9(49)}, c^{41}_{9(50)}, c^{42}_{9(51)}$	
	$c^{61}_{7(52)}, c^{62}_{7(53)}, c^{63}_{7(54)}$	$c^{43}_{9(52)}, c^{44}_{9(53)}, c^{45}_{9(54)}$	

$c^{k}_{i(j)}$ value	1	-1	0
S_{17}	$c^{57}_{1(56)}, c^{60}_{1(59)}, c^{63}_{1(62)}$	$c^{56}_{1(57)}, c^{59}_{1(60)}, c^{62}_{1(63)}$	
	$c^{55}_{2(57)}, c^{58}_{2(60)}, c^{61}_{2(63)}$	$c^{57}_{2(55)}, c^{60}_{2(58)}, c^{63}_{2(61)}$	
	$c^{56}_{3(55)}, c^{59}_{3(58)}, c^{62}_{3(61)}$	$c^{55}_{3(56)}, c^{58}_{3(59)}, c^{61}_{3(62)}$	
	$c^{61}_{4(58)}, c^{62}_{4(59)}, c^{63}_{4(60)}$	$c^{58}_{4(61)}, c^{59}_{4(62)}, c^{60}_{4(63)}$	
	$c^{55}_{5(61)}, c^{56}_{5(62)}, c^{57}_{5(63)}$	$c^{61}_{5(55)}, c^{62}_{5(56)}, c^{63}_{5(57)}$	others
	$c^{58}_{6(55)}, c^{59}_{6(56)}, c^{60}_{6(57)}$	$c^{55}_{6(58)}, c^{56}_{6(59)}, c^{57}_{6(60)}$	
	$c^{37}_{8(55)}, c^{38}_{8(56)}, c^{39}_{8(57)}$	$c^{46}_{7(55)}, c^{47}_{7(56)}, c^{48}_{7(57)}$	
	$c^{40}_{8(58)}, c^{41}_{8(59)}, c^{42}_{8(60)}$	$c^{49}_{7(58)}, c^{50}_{7(59)}, c^{51}_{7(60)}$	
	$c^{43}_{8(61)}, c^{44}_{8(62)}, c^{45}_{8(63)}$	$c^{52}_{7(61)}, c^{53}_{7(62)}, c^{54}_{7(63)}$	

Two qubit self commutation relation:

$c^k_{i(j)}$ value	1	-1	0
S_{22}	$c^3_{10(11)}, c^1_{11(12)}, c^2_{12(10)}$	$c^3_{11(10)}, c^1_{12(11)}, c^2_{10(12)}$	
	$c^3_{13(14)}, c^1_{14(15)}, c^2_{15(13)}$	$c^3_{14(13)}, c^1_{15(14)}, c^2_{13(15)}$	
	$c^3_{16(17)}, c^1_{17(18)}, c^2_{18(16)}$	$c^1_{17(16)}, c^1_{18(17)}, c^2_{16(18)}$	
			others
	$c^6_{10(13)}, c^6_{11(14)}, c^6_{12(15)}$	$c^6_{13(10)}, c^6_{14(11)}, c^6_{15(12)}$	
	$c^4_{13(16)}, c^4_{14(17)}, c^4_{15(18)}$	$c^4_{16(13)}, c^4_{17(14)}, c^4_{18(15)}$	
	$c^5_{16(10)}, c^5_{17(11)}, c^5_{18(12)}$	$c^5_{10(16)}, c^5_{11(17)}, c^5_{12(18)}$	
S_{33}	$c^3_{19(20)}, c^1_{20(21)}, c^2_{21(19)}$	$c^3_{20(19)}, c^1_{21(20)}, c^2_{19(21)}$	
	$c^3_{22(23)}, c^1_{23(24)}, c^2_{24(22)}$	$c^3_{23(22)}, c^1_{24(23)}, c^2_{22(24)}$	
	$c^3_{25(26)}, c^1_{26(27)}, c^2_{27(25)}$	$c^3_{26(25)}, c^1_{27(26)}, c^2_{25(27)}$	
			others
	$c^9_{19(22)}, c^9_{20(23)}, c^9_{21(24)}$	$c^9_{22(19)}, c^9_{23(20)}, c^9_{24(21)}$	
	$c^7_{22(25)}, c^7_{23(26)}, c^7_{24(27)}$	$c^7_{25(22)}, c^7_{26(23)}, c^7_{27(24)}$	
	$c^8_{25(19)}, c^8_{26(20)}, c^8_{27(21)}$	$c^8_{19(25)}, c^8_{20(26)}, c^8_{21(27)}$	
S_{44}	$c^3_{28(29)}, c^1_{29(30)}, c^2_{30(28)}$	$c^3_{28(29)}, c^1_{30(29)}, c^2_{28(30)}$	
	$c^6_{31(32)}, c^4_{32(33)}, c^5_{33(31)}$	$c^6_{32(31)}, c^4_{33(32)}, c^5_{31(33)}$	
	$c^6_{34(35)}, c^4_{35(36)}, c^5_{36(34)}$	$c^6_{35(34)}, c^4_{36(35)}, c^5_{34(36)}$	
			others
	$c^6_{28(31)}, c^6_{29(32)}, c^6_{30(33)}$	$c^6_{31(28)}, c^6_{32(29)}, c^6_{33(30)}$	
	$c^4_{31(34)}, c^4_{32(35)}, c^4_{33(36)}$	$c^4_{34(31)}, c^4_{35(32)}, c^4_{36(33)}$	
	$c^5_{34(28)}, c^5_{35(29)}, c^5_{36(30)}$	$c^5_{28(34)}, c^5_{29(35)}, c^5_{30(36)}$	

Two qubit with two qubit commutation relation:

$c_{i(j)}^{k}$ value	1	-1	0
S_{23}	$c_{10(20)}^{39}, c_{10(23)}^{48}, c_{10(26)}^{57}$	$c_{10(21)}^{38}, c_{10(24)}^{47}, c_{10(27)}^{56}$	others
	$c_{11(21)}^{37}, c_{11(24)}^{46}, c_{11(27)}^{55}$	$c_{11(19)}^{39}, c_{11(22)}^{48}, c_{11(25)}^{57}$	
	$c_{12(19)}^{38}, c_{12(22)}^{47}, c_{12(25)}^{56}$	$c_{12(20)}^{37}, c_{12(23)}^{46}, c_{12(26)}^{55}$	
	$c_{13(20)}^{42}, c_{13(23)}^{51}, c_{13(26)}^{60}$	$c_{13(21)}^{41}, c_{13(24)}^{50}, c_{13(27)}^{59}$	
	$c_{14(21)}^{40}, c_{14(24)}^{49}, c_{14(27)}^{58}$	$c_{14(19)}^{42}, c_{14(22)}^{51}, c_{14(25)}^{60}$	
	$c_{15(19)}^{41}, c_{15(22)}^{50}, c_{15(25)}^{59}$	$c_{15(20)}^{40}, c_{15(23)}^{51}, c_{15(26)}^{60}$	
	$c_{16(20)}^{45}, c_{16(23)}^{54}, c_{16(26)}^{63}$	$c_{16(21)}^{44}, c_{16(24)}^{53}, c_{16(27)}^{62}$	
	$c_{17(21)}^{43}, c_{17(24)}^{52}, c_{17(27)}^{61}$	$c_{17(19)}^{45}, c_{17(22)}^{54}, c_{17(25)}^{63}$	
	$c_{18(19)}^{44}, c_{18(22)}^{53}, c_{18(25)}^{62}$	$c_{18(20)}^{43}, c_{18(23)}^{52}, c_{18(26)}^{61}$	
S_{24}	$c_{10(29)}^{43}, c_{10(32)}^{52}, c_{10(35)}^{61}$	$c_{10(30)}^{40}, c_{10(33)}^{49}, c_{10(36)}^{58}$	others
	$c_{11(29)}^{44}, c_{11(32)}^{53}, c_{11(35)}^{62}$	$c_{11(30)}^{41}, c_{11(33)}^{50}, c_{11(36)}^{59}$	
	$c_{12(29)}^{45}, c_{12(32)}^{54}, c_{12(35)}^{63}$	$c_{12(30)}^{42}, c_{12(32)}^{51}, c_{12(36)}^{60}$	
	$c_{13(30)}^{37}, c_{13(33)}^{46}, c_{13(36)}^{55}$	$c_{13(28)}^{43}, c_{13(31)}^{52}, c_{13(34)}^{61}$	
	$c_{14(30)}^{38}, c_{14(33)}^{47}, c_{14(36)}^{56}$	$c_{14(28)}^{44}, c_{14(31)}^{53}, c_{14(34)}^{62}$	
	$c_{15(30)}^{39}, c_{15(33)}^{48}, c_{15(36)}^{57}$	$c_{15(28)}^{45}, c_{15(31)}^{54}, c_{15(34)}^{63}$	
	$c_{16(28)}^{40}, c_{16(31)}^{49}, c_{16(34)}^{58}$	$c_{16(29)}^{37}, c_{16(32)}^{46}, c_{16(35)}^{55}$	
	$c_{17(28)}^{41}, c_{17(31)}^{50}, c_{17(34)}^{59}$	$c_{17(29)}^{38}, c_{17(32)}^{47}, c_{17(35)}^{56}$	
	$c_{18(28)}^{42}, c_{18(31)}^{51}, c_{18(34)}^{60}$	$c_{18(29)}^{39}, c_{18(32)}^{48}, c_{18(35)}^{57}$	

$c^k_{i(j)}$ value	1	-1	0
S_{34}	$c^{55}_{19(31)}, c^{58}_{19(32)}, c^{61}_{19(33)}$	$c^{46}_{19(34)}, c^{49}_{19(35)}, c^{52}_{19(36)}$	
	$c^{56}_{20(31)}, c^{59}_{20(32)}, c^{62}_{20(33)}$	$c^{47}_{20(34)}, c^{50}_{20(35)}, c^{53}_{20(36)}$	
	$c^{57}_{21(31)}, c^{60}_{21(32)}, c^{63}_{21(33)}$	$c^{48}_{21(34)}, c^{51}_{21(35)}, c^{54}_{21(36)}$	
	$c^{37}_{22(34)}, c^{40}_{22(35)}, c^{43}_{22(36)}$	$c^{55}_{22(28)}, c^{58}_{22(29)}, c^{61}_{22(30)}$	
	$c^{38}_{23(34)}, c^{41}_{23(35)}, c^{44}_{23(36)}$	$c^{56}_{23(28)}, c^{59}_{23(29)}, c^{62}_{23(30)}$	others
	$c^{39}_{24(34)}, c^{42}_{24(35)}, c^{45}_{24(36)}$	$c^{57}_{24(28)}, c^{60}_{24(29)}, c^{63}_{24(30)}$	
	$c^{46}_{25(28)}, c^{49}_{25(29)}, c^{52}_{25(30)}$	$c^{37}_{25(31)}, c^{40}_{25(32)}, c^{43}_{25(33)}$	
	$c^{47}_{26(28)}, c^{50}_{26(29)}, c^{53}_{26(30)}$	$c^{38}_{26(31)}, c^{41}_{26(32)}, c^{44}_{26(33)}$	
	$c^{48}_{27(28)}, c^{51}_{27(29)}, c^{54}_{27(30)}$	$c^{39}_{27(31)}, c^{42}_{27(32)}, c^{45}_{27(33)}$	

Two qubit with three qubit commutation relation:

$c^k_{i(j)}$ value	1	-1	0
S_{25}	$c^{21}_{10(38)}, c^{19}_{11(39)}, c^{20}_{12(37)}$ $c^{21}_{13(41)}, c^{19}_{14(42)}, c^{20}_{15(40)}$ $c^{21}_{16(44)}, c^{19}_{17(45)}, c^{20}_{18(43)}$ $c^{30}_{10(40)}, c^{30}_{11(41)}, c^{30}_{12(42)}$ $c^{28}_{13(43)}, c^{28}_{14(44)}, c^{28}_{15(45)}$ $c^{29}_{16(37)}, c^{29}_{17(38)}, c^{29}_{18(39)}$	$c^{21}_{11(37)}, c^{19}_{12(38)}, c^{17}_{10(39)}$ $c^{21}_{14(40)}, c^{19}_{15(41)}, c^{20}_{13(42)}$ $c^{21}_{17(43)}, c^{19}_{18(44)}, c^{20}_{16(45)}$ $c^{29}_{10(43)}, c^{29}_{11(44)}, c^{29}_{12(45)}$ $c^{30}_{13(37)}, c^{30}_{14(38)}, c^{30}_{15(39)}$ $c^{28}_{16(40)}, c^{28}_{17(41)}, c^{28}_{18(42)}$	others
S_{26}	$c^{24}_{10(47)}, c^{22}_{11(48)}, c^{23}_{12(46)}$ $c^{24}_{13(50)}, c^{22}_{14(51)}, c^{23}_{15(49)}$ $c^{24}_{16(52)}, c^{22}_{17(53)}, c^{23}_{18(54)}$ $c^{33}_{10(49)}, c^{33}_{11(50)}, c^{33}_{12(51)}$ $c^{31}_{13(52)}, c^{31}_{14(53)}, c^{31}_{15(54)}$ $c^{32}_{16(46)}, c^{32}_{17(47)}, c^{32}_{18(48)}$	$c^{24}_{11(46)}, c^{22}_{12(47)}, c^{23}_{10(48)}$ $c^{24}_{14(49)}, c^{22}_{15(50)}, c^{23}_{13(51)}$ $c^{24}_{17(52)}, c^{22}_{18(53)}, c^{23}_{16(54)}$ $c^{32}_{10(52)}, c^{32}_{11(53)}, c^{32}_{12(54)}$ $c^{33}_{13(46)}, c^{33}_{14(47)}, c^{33}_{15(48)}$ $c^{31}_{16(49)}, c^{31}_{17(50)}, c^{31}_{18(51)}$	others
S_{27}	$c^{27}_{10(56)}, c^{25}_{11(57)}, c^{26}_{12(55)}$ $c^{27}_{13(59)}, c^{25}_{14(60)}, c^{26}_{15(58)}$ $c^{27}_{16(62)}, c^{25}_{17(63)}, c^{26}_{18(61)}$ $c^{36}_{10(58)}, c^{36}_{11(59)}, c^{36}_{12(60)}$ $c^{34}_{13(61)}, c^{34}_{14(62)}, c^{34}_{15(63)}$ $c^{35}_{16(55)}, c^{35}_{17(56)}, c^{35}_{18(57)}$	$c^{27}_{11(55)}, c^{25}_{12(56)}, c^{26}_{10(57)}$ $c^{27}_{14(58)}, c^{25}_{15(59)}, c^{26}_{13(60)}$ $c^{27}_{17(61)}, c^{25}_{18(62)}, c^{26}_{16(63)}$ $c^{35}_{10(61)}, c^{35}_{11(62)}, c^{35}_{12(63)}$ $c^{36}_{13(55)}, c^{36}_{14(56)}, c^{36}_{15(57)}$ $c^{34}_{16(58)}, c^{34}_{17(59)}, c^{34}_{18(60)}$	others

$c^k_{i(j)}$ value	1	-1	0
S_{35}	$c^{12}_{19(38)}, c^{15}_{19(41)}, c^{18}_{19(44)}$ $c^{10}_{20(39)}, c^{13}_{20(42)}, c^{16}_{20(45)}$ $c^{11}_{21(37)}, c^{14}_{21(40)}, c^{17}_{21(43)}$ $c^{31}_{25(37)}, c^{32}_{25(40)}, c^{33}_{25(43)}$ $c^{31}_{26(38)}, c^{32}_{26(41)}, c^{33}_{26(44)}$ $c^{31}_{27(39)}, c^{32}_{27(42)}, c^{33}_{27(45)}$	$c^{11}_{19(39)}, c^{14}_{19(42)}, c^{17}_{19(45)}$ $c^{12}_{20(37)}, c^{15}_{20(40)}, c^{18}_{20(43)}$ $c^{10}_{21(38)}, c^{13}_{21(41)}, c^{16}_{21(44)}$ $c^{34}_{22(37)}, c^{35}_{22(40)}, c^{36}_{22(43)}$ $c^{34}_{23(38)}, c^{35}_{23(41)}, c^{36}_{23(44)}$ $c^{34}_{24(39)}, c^{35}_{24(42)}, c^{36}_{24(45)}$	others
S_{36}	$c^{12}_{22(47)}, c^{15}_{22(50)}, c^{18}_{22(53)}$ $c^{10}_{23(48)}, c^{13}_{23(51)}, c^{16}_{23(53)}$ $c^{11}_{24(46)}, c^{14}_{24(49)}, c^{17}_{24(51)}$ $c^{34}_{19(46)}, c^{35}_{19(49)}, c^{36}_{19(52)}$ $c^{34}_{20(47)}, c^{35}_{20(50)}, c^{36}_{20(53)}$ $c^{34}_{21(48)}, c^{35}_{21(51)}, c^{36}_{21(54)}$	$c^{11}_{22(48)}, c^{14}_{22(51)}, c^{17}_{22(54)}$ $c^{12}_{23(46)}, c^{15}_{23(48)}, c^{18}_{23(51)}$ $c^{10}_{24(47)}, c^{13}_{24(50)}, c^{16}_{24(52)}$ $c^{28}_{25(46)}, c^{29}_{25(49)}, c^{30}_{25(52)}$ $c^{28}_{26(47)}, c^{29}_{26(50)}, c^{30}_{26(53)}$ $c^{28}_{27(48)}, c^{29}_{27(51)}, c^{30}_{27(54)}$	others
S_{37}	$c^{12}_{25(56)}, c^{15}_{25(59)}, c^{18}_{25(62)}$ $c^{10}_{26(57)}, c^{13}_{26(60)}, c^{16}_{26(62)}$ $c^{11}_{27(55)}, c^{14}_{27(58)}, c^{17}_{27(60)}$ $c^{28}_{22(55)}, c^{29}_{22(58)}, c^{30}_{22(61)}$ $c^{28}_{23(56)}, c^{29}_{23(59)}, c^{30}_{23(62)}$ $c^{28}_{24(57)}, c^{29}_{24(60)}, c^{30}_{24(63)}$	$c^{11}_{25(57)}, c^{14}_{25(60)}, c^{17}_{25(63)}$ $c^{12}_{26(55)}, c^{15}_{26(57)}, c^{18}_{26(60)}$ $c^{10}_{27(56)}, c^{13}_{27(59)}, c^{16}_{27(61)}$ $c^{31}_{19(55)}, c^{32}_{19(58)}, c^{33}_{19(61)}$ $c^{31}_{20(56)}, c^{32}_{20(59)}, c^{33}_{20(62)}$ $c^{31}_{21(57)}, c^{32}_{21(60)}, c^{33}_{21(63)}$	others

$c^k_{i(j)}$ value	1	-1	0
S_{45}	$c^{16}_{28(40)}, c^{17}_{28(41)}, c^{18}_{28(42)}$	$c^{13}_{28(43)}, c^{14}_{28(44)}, c^{15}_{28(45)}$	
	$c^{10}_{29(43)}, c^{11}_{29(44)}, c^{12}_{29(45)}$	$c^{16}_{29(37)}, c^{17}_{29(38)}, c^{18}_{29(39)}$	
	$c^{13}_{30(37)}, c^{14}_{30(38)}, c^{15}_{30(39)}$	$c^{10}_{30(40)}, c^{11}_{30(41)}, c^{12}_{30(42)}$	
			others
	$c^{22}_{34(37)}, c^{23}_{34(38)}, c^{24}_{34(39)}$	$c^{25}_{31(37)}, c^{26}_{31(38)}, c^{27}_{31(39)}$	
	$c^{22}_{35(40)}, c^{23}_{35(41)}, c^{24}_{35(42)}$	$c^{25}_{32(40)}, c^{26}_{32(41)}, c^{27}_{32(42)}$	
	$c^{22}_{36(43)}, c^{23}_{36(44)}, c^{24}_{36(45)}$	$c^{25}_{33(43)}, c^{26}_{33(44)}, c^{27}_{33(45)}$	
S_{46}	$c^{16}_{31(49)}, c^{17}_{31(50)}, c^{18}_{31(51)}$	$c^{13}_{31(52)}, c^{14}_{31(53)}, c^{15}_{31(54)}$	
	$c^{10}_{32(52)}, c^{11}_{32(53)}, c^{12}_{32(54)}$	$c^{16}_{32(46)}, c^{17}_{32(47)}, c^{18}_{32(48)}$	
	$c^{13}_{33(46)}, c^{14}_{33(47)}, c^{15}_{33(48)}$	$c^{10}_{33(49)}, c^{11}_{33(50)}, c^{12}_{33(42)}$	
			others
	$c^{25}_{28(46)}, c^{26}_{28(47)}, c^{27}_{28(48)}$	$c^{19}_{34(46)}, c^{20}_{34(47)}, c^{21}_{34(48)}$	
	$c^{25}_{29(49)}, c^{26}_{29(50)}, c^{27}_{29(51)}$	$c^{19}_{35(49)}, c^{20}_{35(50)}, c^{27}_{35(51)}$	
	$c^{25}_{30(52)}, c^{26}_{30(53)}, c^{21}_{30(54)}$	$c^{19}_{36(52)}, c^{20}_{36(53)}, c^{21}_{36(54)}$	
S_{47}	$c^{16}_{34(58)}, c^{17}_{34(59)}, c^{18}_{34(60)}$	$c^{13}_{34(61)}, c^{14}_{34(62)}, c^{15}_{34(63)}$	
	$c^{10}_{35(61)}, c^{11}_{35(62)}, c^{12}_{35(63)}$	$c^{16}_{35(55)}, c^{17}_{35(56)}, c^{18}_{35(57)}$	
	$c^{13}_{36(55)}, c^{14}_{36(56)}, c^{15}_{36(57)}$	$c^{10}_{36(58)}, c^{11}_{36(59)}, c^{12}_{36(60)}$	
			others
	$c^{19}_{31(55)}, c^{20}_{31(56)}, c^{21}_{31(57)}$	$c^{22}_{28(55)}, c^{23}_{28(56)}, c^{24}_{28(57)}$	
	$c^{19}_{32(58)}, c^{20}_{32(59)}, c^{27}_{32(60)}$	$c^{22}_{29(58)}, c^{23}_{29(59)}, c^{24}_{29(60)}$	
	$c^{19}_{33(61)}, c^{20}_{33(62)}, c^{21}_{33(63)}$	$c^{22}_{30(61)}, c^{23}_{30(62)}, c^{24}_{30(63)}$	

Three qubit with three qubit commutation relation:

$c^k_{i(j)}$ value	1	-1	0
S_{55}	$c^3_{37(38)}, c^1_{38(39)}, c^2_{39(37)}$ $c^3_{40(41)}, c^1_{41(42)}, c^2_{42(40)}$ $c^3_{43(44)}, c^1_{44(45)}, c^2_{45(43)}$ $c^6_{37(40)}, c^6_{38(41)}, c^6_{39(42)}$ $c^4_{40(43)}, c^4_{41(44)}, c^4_{42(45)}$ $c^5_{43(46)}, c^5_{44(47)}, c^5_{45(48)}$	$c^3_{38(37)}, c^1_{39(38)}, c^2_{37(39)}$ $c^3_{41(40)}, c^1_{42(41)}, c^2_{40(42)}$ $c^3_{44(43)}, c^1_{45(44)}, c^2_{43(45)}$ $c^6_{40(37)}, c^6_{41(38)}, c^6_{42(39)}$ $c^4_{43(40)}, c^4_{44(41)}, c^4_{45(42)}$ $c^5_{37(43)}, c^5_{38(44)}, c^5_{39(45)}$	others
S_{66}	$c^3_{46(47)}, c^1_{47(48)}, c^2_{48(46)}$ $c^3_{49(50)}, c^1_{50(51)}, c^2_{51(49)}$ $c^3_{52(53)}, c^1_{53(54)}, c^2_{54(52)}$ $c^6_{46(49)}, c^6_{47(50)}, c^6_{48(51)}$ $c^4_{49(52)}, c^4_{50(53)}, c^4_{51(54)}$ $c^5_{52(47)}, c^5_{53(48)}, c^5_{54(49)}$	$c^3_{47(46)}, c^1_{48(47)}, c^2_{46(48)}$ $c^3_{50(49)}, c^1_{51(50)}, c^2_{49(51)}$ $c^3_{53(52)}, c^1_{54(53)}, c^2_{52(54)}$ $c^6_{49(46)}, c^6_{50(47)}, c^6_{51(48)}$ $c^4_{52(49)}, c^4_{53(50)}, c^4_{54(51)}$ $c^5_{46(52)}, c^5_{47(53)}, c^5_{48(54)}$	others
S_{77}	$c^3_{55(56)}, c^1_{56(57)}, c^2_{57(55)}$ $c^6_{58(59)}, c^4_{59(60)}, c^5_{60(58)}$ $c^6_{61(62)}, c^4_{62(63)}, c^5_{63(61)}$ $c^6_{55(58)}, c^6_{56(59)}, c^6_{57(60)}$ $c^4_{58(61)}, c^4_{59(62)}, c^4_{60(63)}$ $c^5_{61(55)}, c^5_{62(56)}, c^5_{63(57)}$	$c^3_{56(55)}, c^1_{57(56)}, c^2_{55(57)}$ $c^6_{56(58)}, c^4_{60(59)}, c^5_{58(60)}$ $c^6_{62(61)}, c^4_{63(62)}, c^5_{61(63)}$ $c^6_{58(55)}, c^6_{59(56)}, c^6_{60(57)}$ $c^4_{61(58)}, c^4_{62(59)}, c^4_{63(60)}$ $c^5_{55(61)}, c^5_{56(62)}, c^5_{57(63)}$	others

$c^{k}_{i(j)}$ value	1	-1	0
S_{56}	$c^{62}_{37(51)}, c^{63}_{38(49)}, c^{61}_{39(50)}$ $c^{60}_{37(53)}, c^{58}_{38(54)}, c^{59}_{39(52)}$ $c^{56}_{40(54)}, c^{57}_{41(52)}, c^{55}_{42(53)}$	$c^{63}_{37(50)}, c^{61}_{38(51)}, c^{62}_{39(49)}$ $c^{59}_{37(54)}, c^{60}_{38(52)}, c^{58}_{39(53)}$ $c^{57}_{40(53)}, c^{55}_{41(54)}, c^{56}_{42(52)}$	
	$c^{63}_{40(47)}, c^{61}_{41(48)}, c^{62}_{42(46)}$ $c^{59}_{43(48)}, c^{60}_{44(46)}, c^{58}_{45(47)}$ $c^{57}_{43(50)}, c^{55}_{44(51)}, c^{56}_{45(49)}$	$c^{62}_{40(48)}, c^{63}_{41(46)}, c^{61}_{42(47)}$ $c^{60}_{43(47)}, c^{58}_{44(48)}, c^{59}_{45(46)}$ $c^{56}_{43(51)}, c^{57}_{44(49)}, c^{55}_{45(50)}$	others
	$c^{9}_{37(46)}, c^{9}_{38(47)}, c^{9}_{39(48)}$ $c^{9}_{40(49)}, c^{9}_{41(50)}, c^{9}_{42(51)}$ $c^{9}_{43(52)}, c^{9}_{44(53)}, c^{9}_{45(54)}$		
S_{67}	$c^{62}_{46(60)}, c^{63}_{47(58)}, c^{61}_{48(59)}$ $c^{60}_{46(62)}, c^{58}_{47(63)}, c^{59}_{48(61)}$ $c^{56}_{49(63)}, c^{57}_{50(61)}, c^{55}_{51(62)}$	$c^{63}_{46(59)}, c^{61}_{47(60)}, c^{62}_{48(58)}$ $c^{59}_{46(63)}, c^{60}_{47(61)}, c^{58}_{48(62)}$ $c^{57}_{49(62)}, c^{55}_{50(63)}, c^{56}_{51(63)}$	
	$c^{45}_{49(56)}, c^{43}_{50(57)}, c^{44}_{51(55)}$ $c^{41}_{52(57)}, c^{42}_{53(55)}, c^{40}_{54(56)}$ $c^{39}_{52(59)}, c^{37}_{53(60)}, c^{38}_{54(58)}$	$c^{44}_{49(57)}, c^{45}_{50(55)}, c^{43}_{51(56)}$ $c^{42}_{52(56)}, c^{40}_{53(57)}, c^{41}_{54(55)}$ $c^{38}_{52(60)}, c^{39}_{53(58)}, c^{37}_{54(59)}$	others
	$c^{7}_{46(55)}, c^{7}_{47(56)}, c^{7}_{48(57)}$ $c^{7}_{49(58)}, c^{7}_{50(59)}, c^{7}_{51(60)}$ $c^{7}_{52(61)}, c^{7}_{53(62)}, c^{7}_{54(63)}$		

$c^k_{i(j)}$ value	1	-1	0
S_{75}	$c^{53}_{55(42)}, c^{54}_{56(40)}, c^{52}_{57(41)}$	$c^{54}_{55(41)}, c^{52}_{56(42)}, c^{53}_{57(40)}$	
	$c^{51}_{55(44)}, c^{49}_{56(45)}, c^{50}_{57(43)}$	$c^{50}_{55(45)}, c^{51}_{56(43)}, c^{49}_{57(44)}$	
	$c^{50}_{58(45)}, c^{51}_{59(43)}, c^{49}_{60(44)}$	$c^{51}_{58(44)}, c^{49}_{59(45)}, c^{50}_{60(43)}$	
	$c^{54}_{58(38)}, c^{52}_{59(39)}, c^{53}_{60(37)}$	$c^{53}_{58(39)}, c^{54}_{59(37)}, c^{52}_{60(38)}$	
	$c^{50}_{61(39)}, c^{49}_{62(37)}, c^{51}_{63(38)}$	$c^{51}_{61(38)}, c^{49}_{62(39)}, c^{50}_{63(37)}$	others
	$c^{48}_{61(41)}, c^{46}_{62(42)}, c^{47}_{63(40)}$	$c^{47}_{61(42)}, c^{48}_{62(40)}, c^{46}_{63(41)}$	
	$c^{8}_{55(37)}, c^{8}_{56(38)}, c^{8}_{57(39)}$		
	$c^{8}_{58(40)}, c^{8}_{59(41)}, c^{8}_{60(42)}$		
	$c^{8}_{61(43)}, c^{8}_{62(44)}, c^{8}_{63(45)}$		