UNIVERSITI PUTRA MALAYSIA

PROPERTIES AND COUNTEREXAMPLES ON GENERALIZATIONS OF LINDELOF SPACES

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FSAS 2002 10

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By

ANWAR JABOR ALFAWAKHREH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for Degree of Doctor of Philosophy

February 2002

DEDICATION

To my parents

and

to my brothers and sisters

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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February 2002

Chairman: Adem Kılıçman, Ph.D.

Faculty: Science and Environmental Studies

In this thesis, generalizations of Lindelöf spaces that depend on open covers and regularly open covers are studied. Namely: nearly Lindelöf, almost Lindelöf and weakly Lindelöf spaces. And generalizations of regular-Lindelöf spaces that depend on regular covers are also studied. Namely: nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces. Some properties and characterizations of these six generalizations of Lindelöf spaces are given. The relations among them are studied and some counterexamples are given in order to prove that the studied generalizations are proper generalizations of Lindelöf spaces. Subspaces and subsets of these spaces are studied. We show that some subsets of these spaces inherit these covering properties and some others they do not.

Moreover, semiregular property on these spaces is studied to establish that all of these properties are semiregular properties on the contrary of Lindelöf property. which is not a semiregular property. Mappings and generalized continuous functions are also studied on these generalizations and we prove that these properties are topological properties. Relations and some properties of many decompositions of continuity and generalized continuity that recently defined and studied are given. Counterexamples are also given to establish the relations among these generalizations of continuity. We show that some proper mappings preserve these topological properties such as: δ-continuity preserves nearly Lindelöf property. θ-continuity preserves almost Lindelöf property. R-maps preserve nearly regular-Lindelöf property. Almost continuity preserves weakly Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf properties. Moreover, we give some conditions on the functions or on the spaces to prove that weak forms of continuity preserve some of these covering properties under these conditions.

The product property on these generalizations is also studied. We show that these topological properties, as in the case of most non-compact properties, are not preserved by product, even under a finite product. Some conditions are given on these generalizations to prove that these properties are preserved by finite product under these conditions. We show that, in weak P-spaces, finite product of nearly Lindelöf spaces is nearly Lindelöf and finite product of weakly Lindelöf spaces is almost Lindelöf.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

SIFAT-SIFAT DAN CONTOH-CONTOH BERKAITAN DALAM PENGITLAKAN BAGI RUANG LINDELÖF

Oleh

ANWAR JABOR ALFAWAKHREH

Februari 2002

Pengerusi: **Adem Kılıçman, Ph.D.**

Fakulti: **Sains dan Pengajian Alam Sekitar**

Dalam tesis ini, pengitlakan untuk ruang Lindelöf yang bergantung kepada tudung terbuka dan tudung terbuka sekata telah di kaji. Beberapa contoh adalah: hampir Lindelöf, dekat Lindelöf dan ruang lemah Lindelöf. Pengitlakan untuk ruang Lindelöf-sekata yang bergantung kepada tudung sekata juga telah dikaji. Beberapa contoh adalah: hampir Lindelöf-sekata, dekat Lindelöf-sekata dan ruang lemah Lindelöf-sekata. Beberapa sifat-sifat dan ciri-ciri untuk kesemua enam pengitlakan bagi ruang Lindelöf telah diberi. Perhubungan di antara mereka telah dikaji dan beberapa contoh-contoh yang berkaitan telah diberi untuk membuktikan bahawa pengitlakan yang telah dikaji adalah merupakan pengitlakan yang sepadan bagi ruang Lindelöf. Subruang dan subset bagi semua ruang ini telah dikaji. Kita akan tunjukkan bahawa setengah subset bagi ruang-ruang ini mewarisi sifat-sifat yang menyeluruh ini dan setengah lagi tidak.

Selain daripada itu, sifat sepera-sekata bagi ruang-ruang ini juga telah dikaji untuk membentuk bahawa kesemua sifat-sifat adalah sepera-sekata dan

adalah bertentangan dengan sifat-sifat bagi Lindelöf, yang bukan merupakan sifat-sifat sepera-sekata. Pemadanan dan fungsi selanjar teritlak juga telah dikaji bagi ruang-ruang tersebut untuk membuktikan bahawa sifat-sifat tersebut adalah sifat-sifat topologi. Perhubungan dan beberapa sifat-sifat bagi banyak dekompososi bagi keselanjaran dan keselanjaran teritlak yang baru ditakrif dan dikaji telah diberi. Beberapa contoh-contoh berkaitan telah diberi untuk membentuk perhubungan diantara pengitlakan bagi keselanjaran. Kita telah tunjukkan bahawa pemadanan yang sewajarnya mengekalkan sifat-sifat topologi ini seperti: keselanjaran-δ mengekalkan sifat-sifat hampir Lindelof. keselanjaran-θ mengekalkan sifat-sifat dekat Lindelöf. Peta-R mengekalkan sifat-sifat hampir Lindelöf-sekata. Dekat selanjaran mengekalkan sifat-sifat lemah Lindelöf, dekat Lindelöf-sekata dan lemah Lindelöf-sekata. Selain daripada itu, kita memberikan beberapa syarat-syarat kepada fungsi ataupun ruang untuk membuktikan bahawa bentuk-bentuk yang lemah bagi keselanjaran mengekalkan beberapa sifat-sifat menyeluruh dibawah syarat-syarat ini.

Topologi hasil darab Tychonoff bagi pengitlakan ini juga telah dikaji. Kita dapat menunjukkan bahawa sifat-sifat Topologi ini , seperti dalam kebanyakkan sifat tak padat, adalah tidak kekal oleh topologi hasil darab walaupun di bawah hasil darab t'hingga. Beberapa syarat-syarat telah diberi bagi pengitlakan untuk mebuktikan bahawa sifat-sifat tersebut adalah kekal oleh topologi hasil darab t'hingga dibawah syarat-syarat ini. Kita telah menunjukkan bahawa, dalam ruang P-lemah, hasil darab t'hingga [illegible] hampir Lindelöf dan hasil darab t'hingga [illegible] dekat Lindelöf.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor Associate Professor Dr. Adem Kılıçman for his guidance, supervision and encouragement during my study. I am also grateful to the members of the supervisory committee Associate Professor Dr. Ural Bekbaev and Associate Professor Dr. Mat Rofa Ismail for their comments, advices and kindness. My thanks go also to [illegible] Budin for his comments on this thesis. I am also grateful to the independent examiner Professor Dr. Lawrence Michael Brown for his valuable comments on this thesis.

I would also like to express my gratitude to all my friends and colleagues in UPM for their moral support and encouragement, especially to Majdi Al-qdah for correcting the English grammar in this thesis. My sincere gratitude goes also to my best friends in UPM Raed Alkhasawneh and Jayanthi Arasan.

I certify that an Examination Committee met on 21st February 2002 to conduct the final examination of Anwar Jabor Alfawakhreh on his Doctor of Philosophy thesis entitled "Properties and Counterexamples on Generalizations of Lindelöf Spaces" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

ANWAR JABOR ALFAWAKHREH

Date: 14/4/2002

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LIST OF ABBREVIATIONS

Int(A) or $\mathrm{Int}_X(A)$	Interior of A in X
Cl(A) or $\mathrm{Cl}_X(A)$	Closure of A in X
Bd(A)	Boundary of A
$\mathbb{R}$	The set of Real numbers
$\mathbb{Q}$	The set of Rational numbers
$\mathbb{P}$	The set of Irrational numbers
$\mathbb{N}$	The set of Integer numbers
$X \setminus A$	The Complement of A in X
(X, τ)	Topological Space X
(X, τ^*)	The Semiregularization topology of (X, τ)
τ_A	The Induced Topology on A
Ω	First Uncountable Ordinal
$\aleph_0$	Cardinality of $\mathbb{N}$
$\emptyset$	The empty Set
$\subseteq$	Subset
τ_{dis}	The discrete topology
τ_{ind}	The indiscrete topology
$\tau_{l.r.}$	The left ray topology
τ_{cof}	The cofinite (finite complement) topology
τ_{coc}	The cocountable (countable complement) topology
τ_u	The Usual (Euclidean) topology
τ_s, E	Sorgenfrey line space
$E \times E$	The Sorgenfrey plane space
$\|A\|$	Cardinal of A

CHAPTER 1

INTRODUCTION

In this chapter we give some historical remarks on generalizations of Lindelöf and regular-Lindelöf spaces. Basic definitions and theorems that are used in other chapters are also given. Before starting, we recall that spaces mean topological spaces on which no separation axioms are assumed unless explicitly stated otherwise, and the topological space (X, τ) will be replaced by X if there is no chance for confusion. The symbols $\mathbb{N}, \mathbb{R}, \mathbb{P}$ and $\mathbb{Q}$ are used to denote the natural numbers, real numbers, irrational numbers and rational numbers, respectively. Ω denotes the first uncountable ordinal number.

1.1 Historical Remarks

A topological space (X, τ) is said to be Lindelöf if every open cover of the space X has a countable subcover. The Lindelöf theorem, that every second countable space is Lindelöf, was proved for Euclidean spaces as early as 1903 by Ernest Leonard Lindelöf (1870 - 1946) (*Sur Quelques point de la théorie des Ensembles*). The formal study of Lindelöf spaces was begun in 1921 by Kuratowski and Sierpinski (*La Théoréme de Boral-Lebesgue dans la Théorie des Ensembles Abstraits*). Lindelöf spaces were called finally compact spaces by authors in Russia, see for example, Alexandroff (*Some results in the theory of topological spaces*) (Willard, 1970).

The idea of Lindelöf property came from studying compactness property. Since compactness is a very important property in topology and analysis, mathematicians have studied it widely. In compact spaces we deal with open covers to get finite subcovers. After that compactness was generalized to countable compactness. The idea of Lindelöf property came later by dealing with open covers to get countable subcovers.

Since the relationship between compactness and Lindelöf property is very strong, where every compact space is Lindelöf but not conversely, many properties of compact spaces were generalized onto Lindelöf spaces. Thus mathematicians first studied some generalizations of compact spaces as: nearly compactness, paracompactness, realcompactness, weakly compactness, metacompactness, then they generalized these concepts onto Lindelöf spaces and they studied nearly Lindelöf, almost Lindelöf, weakly Lindelöf, paralindelöf and other generalizations. In fact, most of these generalizations of compact spaces or Lindelöf spaces depend on one or both of the following:
(1) The given covers that we deal with as: open cover, closed cover, regularly open cover, regularly closed cover, regular cover, countable open cover.
(2) The resulting subfamily of the given cover that we get after applying the property as: has a finite subcover, has a countable subcover, has irreducible subcover, has a locally finite refinement, has a point finite refinement, has a star-refinement, has a countable subfamily whose union of its elements is dense, has a countable subfamily whose closures of its elements is a cover or the interior of the closure of the elements is a cover. All of these generalizations are related to compact property or Lindelöf property and have given a name derived from the two words "compact" or "Lindelöf".

Many generalizations of Lindelöf spaces have been introduced and several authors for different reasons and purposes studied these generalizations of Lindelöf spaces. Some of these generalizations depend by their definitions on open covers as in Lindelöf spaces. Others depend on regularly open covers, which are covers by regularly open sets in the topological space. Some others depend on regularly closed covers, that are covers by regularly closed sets in the space and some other generalizations depend on regular covers which are introduced by Cammaroto and Lo Faro (1981).

As a generalizations of Lindelöf spaces that depend on open covers and regularly open covers, Frolik (1959) introduced the notion of weakly Lindelöf spaces, that afterward were studied by many authors, see for examples, (Comfort et al, 1969), (Hager, 1969), (Ulmer, 1972), (Woods, 1976) and (Bell et al., 1978). About this topic, Balasubramanian (1982) introduced and studied nearly Lindelöf spaces that are between Lindelöf spaces and weakly Lindelöf spaces. Willard and Dissanayake (1984) gave the notion of almost k-Lindelöf spaces, that for $k = \aleph_0$ are called almost Lindelöf spaces, and that are between nearly Lindelöf spaces and weakly Lindelöf spaces.

Moreover, Cammaroto and Santoro (1996) studied some properties and relations among nearly Lindelöf, almost Lindelöf and weakly Lindelöf spaces. It was also introduced the notions of almost regular-Lindelöf, weakly regular-Lindelöf and nearly regular-Lindelöf spaces as a new generalizations of Lindelöf spaces that depend on regular covers. Cammaroto and Santoro have studied some properties of almost regular-Lindelöf spaces and they left the study of the other generalizations open, see (Cammaroto and Santoro, 1996).

1.2 Regularly Open Sets

Let (X, τ) be a topological space and $A \subseteq X$, the interior of A in X is the union of all subsets of X that are contained in A, in symbols,

$$\mathrm{Int}(A) = \cup\{G \subseteq X : G \in \tau \text{ and } G \subseteq A\}.$$

The closure of A in X is defined in the following way.

$$\mathrm{Cl}(A) = \cap\{K \subseteq X : K \text{ is closed and } A \subseteq K\}.$$

In this thesis we use the notations $\mathrm{Int}_\tau(A)$ or $\mathrm{Int}_X(A)$ to mean $\mathrm{Int}(A)$ and the notations $\mathrm{Cl}_\tau(A)$ or $\mathrm{Cl}_X(A)$ to mean $\mathrm{Cl}(A)$ in (X, τ) where confusion is possible as to what space are the interior and the closure to be taken in. The notion of interior and closure are dual to each other, as open and closed are. The nature of this duality can be observed by the equations, $X \backslash \mathrm{Int}(A) = \mathrm{Cl}(X \setminus A)$ and $X \setminus \mathrm{Cl}(A) = \mathrm{Int}(X \setminus A)$. Note also that by a clopen subset of X we mean an open and closed subset in X.

Definition 1.2.1 An open subset $A \subsetneq X$ is regularly open iff A is the interior of its closure (i.e. $A = \mathrm{Int}(\mathrm{Cl}(A))$). A closed subset $F \subseteq X$ is regularly closed iff F is the closure of its interior (i.e. $F = \mathrm{Cl}(\mathrm{Int}(F))$).

Note that in any topological space X, $\emptyset$ and X are regularly open and regularly closed. The complement of a regularly open set is regularly closed and vice versa. If $A \subsetneq X$ then $\mathrm{Int}(\mathrm{Cl}(A))$ is regularly open in X and $\mathrm{Cl}(\mathrm{Int}(A))$ is regularly closed in X. Recall also that if (X, τ) is a topological space, then a base for τ is a collection $\mathcal{B} \subset \tau$ for which every open subset of X is a union of elements of $\mathcal{B}$. Evidently, $\mathcal{B}$ is a base for X iff whenever G is an open set in X and $p \in G$, then there is some $B \in \mathcal{B}$ such that $p \in B \subseteq G$.

Example 1.2.1 Let X be any set. On X define the following topologies.

(i) The discrete topology, denoted by (X, τ_{dis}), to be the collection of all subsets of X.

(ii) The indiscrete topology, denoted by (X, τ_{ind}), as follows $\tau_{ind} = \{\emptyset, X\}$.

(iii) The complement finite topology (or cofinite topology), denoted by (X, τ_{cof}), as follows $\tau_{cof} = \{U \subset X : X \setminus U$ is finite $\} \cup \{\emptyset, X\}$.

(iv) The complement countable topology (or cocountable topology), denoted by (X, τ_{coc}), as follows $\tau_{coc} = \{U \subset X : X \setminus U$ is countable $\} \cup \{\emptyset, X\}$.

(v) The Euclidean (or usual) topology on the set of real numbers $\mathbb{R}$, denoted by $(\mathbb{R}, \tau_u)$, by taking basis sets of the form $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

(vi) The left ray topology (or left order topology) on $\mathbb{R}$, denoted by $(\mathbb{R}, \tau_{l.r.})$, by taking basis sets of the form $P_a = \{x \in \mathbb{R} : x < a\}$.

(v) The Sorgenfrey line, denoted by $(\mathbb{R}, \tau_{sor})$ or E, is the set of real numbers $\mathbb{R}$ with the topology generated by half-open intervals $[a, b) = \{x \in \mathbb{R} : a \leq x < b; a, b \in \mathbb{R}\}$.

Now, in the discrete topology (X, τ_{dis}) for any set X, all subsets of X are regularly open and regularly closed. In the indiscrete topology (X, τ_{ind}), $\emptyset$ and X are regularly open and regularly closed. In the usual topology (Euclidean space), $(\mathbb{R}, \tau_u)$, the open sets (a, b) are regularly open and the closed sets $[a, b]$ are regularly closed for any points $a, b \in \mathbb{R}$ with $a < b$. In the Sorgenfrey line E, the basic open sets $[a, b)$ are regularly open and regularly closed. However, in the left ray topology $(\mathbb{R}, \tau_{l.r.})$, the complement finite topology $(\mathbb{R}, \tau_{cof})$ and the complement countable topology $(\mathbb{R}, \tau_{coc})$ on the set of real numbers $\mathbb{R}$, only $\emptyset$ and $\mathbb{R}$ are regularly open and regularly closed.

We also note that arbitrary union of open sets is open, arbitrary intersection of closed sets is closed, finite intersection of open sets is open and finite union of closed sets is closed. But this is not true, in general, for regularly open and regularly closed sets. In fact the intersection, but not necessarily the union, of two regularly open sets is regularly open. The union, but not necessarily the intersection, of two regularly closed sets is regularly closed. That is, finite union of regularly closed sets is regularly closed and finite intersection of regularly open sets is regularly open. Note that in the usual topology, $(\mathbb{R}, \tau_u)$, the union of the regularly open sets $(1,2)$ and $(2,3)$ is $(1,2) \cup (2,3)$, which is not regularly open in $(\mathbb{R}, \tau_u)$. The intersection of the regularly closed sets $[1,2]$ and $[2,3]$ is $\{2\}$, which is not regularly closed in $(\mathbb{R}, \tau_u)$ since $\mathrm{Cl}(\mathrm{Int}(\{2\})) = \emptyset$. The following proposition gives a characterization for regularly open and regularly closed sets. Its proof is easy and is not given here.

Proposition 1.2.1 Let (X, τ) be a topological space and $A \subseteq X$ then,
(a) A is regularly open if and only if there exists a closed subset F of X such that $A = \mathrm{Int}(F)$.
(b) A is regularly closed if and only if there exists an open subset G of X such that $A = \mathrm{Int}(G)$.

Next we state the definition of P-spaces and weak P-spaces that will be used in the next chapters.

Definition 1.2.2 (i) A space X is called a P-space iff countable intersection of open sets is open. (ii) A space X is called a weak P-space (Mukherji and Sarkar, 1979) if for each countable family $\{U_n : n \in \mathbb{N}\}$ of open sets in X, we have $\mathrm{Cl}(\bigcup_{n \in \mathbb{N}} U_{\alpha_n}) = \bigcup_{n \in \mathbb{N}} \mathrm{Cl}(U_{\alpha_n})$.

Note that a space X is a P-space iff countable union of closed sets is closed. Moreover, X is a weak P-space iff countable intersection of regularly open sets is regularly open iff countable union of regularly closed sets is regularly closed.

Example 1.2.2 The complement finite topology on the set of real numbers $\mathbb{R}$ is not a P-space since $\{U_n = \mathbb{R}\backslash\{1, \ldots, n\} : n \in \mathbb{N}\}$ is a countable family of open sets but $\bigcap_{n\in\mathbb{N}} U_n = \mathbb{R}\backslash\{1, \ldots, n, \ldots\}$ which is not open. But the complement finite topology on $\mathbb{R}$ is a weak P-space since the only regularly open sets are $\emptyset$ and $\mathbb{R}$ itself which their intersection is regularly open. Moreover, the complement countable topology on the set of real numbers $\mathbb{R}$ is a P-space since if $\{U_n : n \in \mathbb{N}\}$ is a countable family of open sets in $(\mathbb{R}, \tau_{coc})$, then $U_n = \mathbb{R}\backslash G_n$ where G_n is a countable subset of $\mathbb{R}$ for every $n \in \mathbb{N}$. Thus $\bigcap_{n\in\mathbb{N}} U_n = \mathbb{R}\backslash \bigcup_{n\in\mathbb{N}} G_n$ which is open in $(\mathbb{R}, \tau_{coc})$ since countable union of countable sets is countable. It is also clear that $(\mathbb{R}, \tau_{coc})$ is a weak P-space.

Recall that a space X is called extremally disconnected if the closure of open sets in X is open or, equivalently, the interior of closed sets is closed. It is obvious that in extremally disconnected spaces, regularly open sets and regularly closed sets are clopen (i.e. closed and open). Moreover, a space X is called submaximal if every dense subset of X is open.

1.3 Semiregular Spaces

It was pointed that, in any topological space (X, τ), the intersection of two regularly open sets is regularly open. So the regularly open sets in any topological space (X, τ) form a base for a coarser topology called the semiregularization of (X, τ), which is given in the following definition.

Definition 1.3.1 The topology generated by the regularly open sets in the topological space (X, τ) is called the semiregularization topology of (X, τ) and is denoted by (X, τ^*).

It is clear that $\tau^* \subseteq \tau$, but τ is not necessary a subset of τ^*. If, in the other hand, $\tau^* = \tau$ then this yields semiregular spaces.

Definition 1.3.2 The topological space (X, τ) is called semiregular if the regularly open sets form a base for the topology. Or equivalently, $\tau^* = \tau$.

Example 1.3.1 For any set X, the basic open sets in (X, τ_{dis}), (X, τ_{ind}), $(\mathbb{R}, \tau_{sor})$ and $(\mathbb{R}, \tau_u)$ spaces are all regularly open. Thus the regularly open sets in these topological spaces generate the same spaces. [illegible] $= (X, \tau_{dis})$, $(X, \tau^*_{ind}) = (X, \tau_{ind})$, $(\mathbb{R}, \tau^*_{sor}) = (\mathbb{R}, \tau_{sor})$ and $(X, \tau^*_u) = (X, \tau_u)$, which implies that all these spaces are semiregular. However, the regularly open sets in $(\mathbb{R}, \tau_{cof})$, $(\mathbb{R}, \tau_{coc})$ and $(\mathbb{R}, \tau_{l.r.})$ spaces are only $\emptyset$ and $\mathbb{R}$ which generate the indiscrete topology $(\mathbb{R}, \tau_{ind})$. So $(\mathbb{R}, \tau^*_{cof}) = (\mathbb{R}, \tau^*_{coc}) = (\mathbb{R}, \tau^*_{l.r.}) = \{\emptyset, \mathbb{R}\}$. This implies that $(\mathbb{R}, \tau_{cof})$, $(\mathbb{R}, \tau_{coc})$ and $(\mathbb{R}, \tau_{l.r.})$ are not semiregular spaces.

Recall that, a topological space X is called a regular space iff, for any closed set F in X and any singleton $\{x\}$ disjoint from F, there exist two disjoint open sets U and V such that $F \subseteq U$ and $x \in V$. Equivalently, X is regular iff, for each $x \in X$ and each open set U_x in X containing x, there exists an open set V_x containing x such that $x \in V_x \subseteq \mathrm{Cl}(V_x) \subseteq U_x$. It is clear that every regular space is semiregular but the converse is not true, in general, as the following example shows.

Example 1.3.2 Let $X = [-1, 1] \subset \mathbb{R}$ and $\tau = \{U \subseteq X : 0 \notin U \text{ or } (-1, 1) \subseteq U\}$. Clearly, if $0 \in A \subseteq X$ then A is closed. Now, if $U \in \tau$ then we have three cases.

(i) $0 \notin U$ and U is not closed. In this case $\mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Int}(U \cup \{0\}) = U$.

Thus $U \in \tau^*$.

(ii) $0 \notin U$ and U is closed. In this case $\mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Int}(U \cup \{0\}) = U$. Thus U is clopen and hence $U \in \tau^*$.

(iii) $(-1, 1) \subseteq U$. So U is clopen and hence $\mathrm{Int}(\mathrm{Cl}(U)) = U$. Thus $U \in \tau^*$.

This implies that $\tau \subseteq \tau^*$. Hence $\tau^* = \tau$. Therefore X is semiregular. However, X is not regular since $\{0\}$ is a closed set and $\frac{1}{2} \notin \{0\}$ but for any open set U containing $\{0\}$ it contains $(-1, 1)$. Hence $\frac{1}{2} \in U$. Thus U intersects any open set containing $\frac{1}{2}$.

The following propositions give some properties of semiregular spaces. Their proofs are relatively easy so omitted.

Proposition 1.3.1 If A and B are two disjoint open sets in (X, τ), then $\mathrm{Int}(\mathrm{Cl}(A))$ and $\mathrm{Int}(\mathrm{Cl}(B))$ are disjoint basic open sets in (X, τ^*) containing A and B, respectively.

Proposition 1.3.2 Let (X, τ) be a topological space and $A \subseteq X$ be open or dense in (X, τ). Then (a) The regularly open (resp. regularly closed) sets in the induced topology (A, τ_A) are of the form $V \cap A$ where V is regularly open (resp. regularly closed) in (X, τ).

(b) $(\tau_A)^* = {\tau^*}_A$.

Proposition 1.3.3 Let (X, τ) be a topological space and (X, τ^*) its semiregularization. Then

(a) Regularly open sets of (X, τ) are the same as the regularly open sets of (X, τ^*).

(b) Regularly closed sets of (X, τ) are the same as the regularly closed sets of

(X, τ^*).

(c) Clopen sets of (X, τ) are the same as the clopen sets of (X, τ^*).

(d) $\mathrm{Cl}_\tau(U) = \mathrm{Cl}_{\tau^*}(U)$ for every $U \in \tau$.

(e) $(\tau^*)^* = \tau^*$.

Note that extremally disconnected property and semiregular property are independent properties. The usual topology $(\mathbb{R}, \tau_u)$ is semiregular but not extremally disconnected. The following example is extremally disconnected but not semiregular, first we need to define the following concepts.

A filter on a set X is a collection F of subsets of X with the following properties. (i) Every subset of X contains a set of F belongs to F. (ii) Every finite intersection of sets of F belongs to F. (iii) The empty set $\emptyset$ is not in F.

Recall also that if a filter F on X has the property that there is no filter on X which is strictly finer than F, F is called an ultrafilter on X. If a point x is in all the sets of a filter then we call it a cluster point of F. An ultrafilter with no cluster point is called nonprincipal.

Example 1.3.3 (Steen and Seebach, 1978). Let $\mathbb{Z}^+$ be the set of positive integers, and let M be the collection of all nonprincipal ultrafilters on $\mathbb{Z}^+$. Let $X = \mathbb{Z}^+ \cup M$, and let the topology τ on X be generated by the points of $\mathbb{Z}^+$ together with all sets of the form $A \cup \{F\}$ where $A \in F \in M$. Then the space X is extremally disconnected and to show this we prove that for any open subset G of X we have $\mathrm{Cl}(G)$ is open. Suppose that p is a limit point of G which does not belong to G; since each point of $\mathbb{Z}^+$ is open, $p \in X \setminus \mathbb{Z}^+ = M$. So p is an ultrafilter, say F, and every neighborhood $A \bigcup \{F\}$ of p (where $A \in F$) intersects G. But since F itself does not belong to G, this intersection