UNIVERSITI PUTRA MALAYSIA

COMPUTATION OF MAASS CUSP FORMS ON SINGLY PUNCTURED TWO-TORUS AND TRIPLY PUNCTURED TWO-SPHERE USING MATHEMATICA

ABUBAKER AHMED MOHAMED SIDDIG

IPM 2009 8

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By

ABUBAKER AHMED MOHAMED SIDDIG

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

October 2009

DEDICATION

To my parents, my brothers , my sisters and my wife

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

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Chairman : Associate Professor Hishamuddin Zainuddin, PhD

Institute : Institute For Mathematical Research

The topic of this study is the computation of Maass cusp form, i.e. the eigenfunctions of the hyperbolic Laplace-Beltrami operator on punctured surfaces namely singly punctured two-torus and triply punctured two-sphere. Punctured surfaces are surfaces with points removed or located infinitely far away and they have complex topological and geometrical properties. The presence of the punctures or cusps means that there is a continuous spectrum as well as the discrete one. This work focuses on the discrete part of computational nature.

Hejhal developed an algorithm to compute Maass cusp form on triangle groups. The algorithm of Hejhal is based on automorphy condition and also applies to the computation of the Maass cusp forms on Fuchsian group whose the fundamental domain has exactly one cusp.

In this work the method due to Hejhal was recalled and extended for computation of Maass cusp on singly punctured two-torus which still has one cusp but a nonzero genus. The algorithm was modified further to carry out the computation for the surfaces with three cusps i.e. triply punctured two-sphere. All the computations were implemented in Mathematica and built in a way accessible to any one with an introductory knowledge in Mathematica.

The results of the study are the first low-lying eigenvalues, examples of Fourier coefficients and graphic plots of Maass cusp forms each for modular group, singly punctured two-torus and triply punctured two-sphere. The eigenvalues and the Fourier coefficients were computed with the desired accuracy. Some comparisons between singly punctured two-torus and triply punctured two-sphere are also presented.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KOMPUTASI FUNGSI BENTUK JURING MAASS DI ATAS 2-TORUS BERJURING TUNGGAL DAN 2 SFERA BERJURING TIGA MENGGUNAKAN MATHEMATICA

Oleh

ABUBAKER AHMED MOHAMED SIDDIG

Oktober 2009

Pengerusi : Profesor Madya Hishamuddin Zainuddin, PhD

Institut : Institut Penyelidikan Matematik

Topik kajian ini adalah pengiraan fungsi bentuk juring Maass, yang merupakan fungsi eigen operator hiperbolik Laplace-Beltrami di atas permukaan berjuring iaitu 2- torus berjuring tunggal dan 2-sfera berjuring tiga. Permukaan berjuring adalah permukaan dengan beberapa titik diasingkan atau diletakkan pada jarak infiniti dan permukaan seumpama ini mempunyai ciri topologi dan geometri yang kompleks. Kewujudan juring bermakna terdapat kedua-dua spektrum yang selanjar dan yang diskrit. Hejhal telah membangunkan satu algoritma untuk mengira fungsi bentuk juring Maass untuk kumpulan segi tiga berdasarkan syarat automorfi dan turut dapat digunakan untuk mengira fungsi bentuk juring Maass untuk kumpulan Fuchsian, yang domain asasnya mempunyai satu juring.

Dalam kajian ini, kaedah penyelesaian Hejhal diguna kembali dan diperluaskan kepada pengiraan fungsi bentuk juring Maass bagi 2-torus berjuring tunggal yang masih mempunyai satu juring tetapi bergenus bukan sifar. Algoritma ini kemudian diubah seterusnya untuk membuat pengiraan bagi permukaan yang mempunyai tiga juring, iaitu 2 sfera berjuring tiga. Semua pengiraan dibuat menggunakan Mathematica dan dibangun dengan cara yang boleh digunakan oleh sesiapa sahaja yang mempunyai pengetahuan awal Mathematica.

Hasil peyelidikan ini adalah nilai eigen paras rendah, contoh pekali Fourier dan plot grafik fungsi bentuk juring Maass untuk kumpulan modular, 2-torus berjuring tunggal dan 2-sfera berjuring tiga. Nilai eigen dan pekali Fourier telah dikira mengikut kejituan yang diingini. Perbandingan juga dibuat antara 2-torus berjuring tunggal dengan 2 sfera berjuring tiga.

ACKNOWLEDGEMENT

In the name of Allah, the Beneficent, the Most Merciful. Praises and thanks be to Allah, The Lord of the Worlds, and peace be upon the last messenger, Mohammed, his Family an Companions.

This thesis would not exist without the support and generosity of my advisor,
Assoc. Prof. Dr. Hishamuddin Zainuddin. I have learned a great deal from working with him, and it is a pleasure to thank him now for making it such a pleasant experience; many of the ideas that have been developed here are due to him. I am most grateful to Holger Then for his friendly advice and valuable discussions in carrying this numerical work. I am also grateful to Prof. Adem Kilickman and Assoc. Prof. Dr. Jumiah for their support. I would like to thank all my dear friends. I would rather not list their names here, but merely trust that they know who I am referring to should they ever read this.

My sincere appreciation goes to my parents, brothers and sisters and my wife for all their years of unconditional love and support, without which I would not be here now. I am obliged to Hassan Eltayeb for encouraging me to use the LaTeX as platform to write this manuscript. I am indebted to my friend Ustaz Kamaruddin bin Dauod for his prayer and support, may Allah bless him.
This work is supported by the Malaysian Ministry of Science, Technology and Innovations (MOSTI), under the project no. 06-01-04-SF0256. I would like to thank the Institute of Advanced Technology, Universiti Putra Malaysia where the project is initiated.

Finally I pray to Almighty Allah to reward all those who contributed, either directly or indirectly, to the success of this work.

I certify that a Thesis Examination Committee has met on **23 October 2009** to conduct the final examination of **Abubaker Ahmed Mohamed Siddig** on his thesis entitled " **COMPUTATION OF MAASS CUSP FORMS ON SINGLY PUNCTURED TWO-TORUS AND TRIPLY PUNCTURED TWO-SPHERE USING MATHEMATICA** " in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the (Doctor of Philosophy).

Members of the Examination Committee are as follows:

Abdul Halim Shaari, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Zuriati Ahmad Zukarnain, PhD
Lecturer
Faculty of Computer Science and Information Technology
Universiti Putra Malaysia
(Internal Examiner)

Zainal Abidin Talib, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Kurunathan a/l Ratnavelu, PhD
Professor
Institute Sains Matematik
Universiti Malaya
(External Examiner)

Bujang Kim Huat, PhD
Professor/Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

This thesis submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfillment of the requirement for the degree of Doctor of Philosophy. The members of Committee are as follows:

Hishamuddin Zainuddin, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Adem Kilicman, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Jumiah Hassan, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

HASANAH MOHD. GHAZALI, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

ABUBAKER AHMED MOHAMED SIDDIG

Date: 9 November 2009

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LIST OF ABBREVIATIONS

Z	Integers
P	Real number
Θ	Rational number
X	Complex number
∞	Infinity
H	The upper half- plane
Γ	The Modular group
$\Gamma(2)$	Principal congruence subgroup of level 2
G	Finite 3-regular graph
O	Orientation
$S^{O}(G, O)$	Finite-area Riemann surface
$S^{O}(G, O)$	Closed Riemann surface
LHT	Left hand turn
χ	Euler Characteristic
$\Gamma(N)$	Principal congruence subgroup of level N
Δ	Hyperbolic Laplace operator
λ	Eigenvalue
K_{ir}	K-Bessel function with imaginary order
SL(2, Z)	Special Linear Two-Dimensioanl of integers
SL(2, P)	Special Linear Two-Dimensioanl of reals
PSL(2, Z)	Projective Special Linear Two-Dimensioanl of integers
F	Fundamental domain

$F_{i\infty}$	Fundamental domain with a cusp at $z=i\infty$
F_0	Fundamental domain with a cusp at $z=0$
F_1	Fundamental domain with a cusp at $z=1$
F^{I}	Subdomain with a cusp at $z=i\infty$
F^{S}	Subdomain with a cusp at $z=0$
$F^{ST^{-1}}$	Subdomain with a cusp at $z=1$

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MANUSCRIPT UNDER PREPRATION

Abubaker Ahmed Mohamed Siddig and Hishamuddin Zainuddin, A Numerical Study of Eigenvalues of the Hyperbolic Laplacian for Singly Punctured Two-torus.

Abubaker A. M. Siddig and Hishamuddin Zainuddin, Maass Cusp Forms on Triply Punctured Two-Sphere.