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Pursuit Differential Game Described by Infinite First Order 2-Systems of Differential Equations

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ABSTRACT

We study a pursuit differential game problem for infinite first order 2-systems of differential equations in the Hilbert space l_2. Geometric constraints are imposed on controls of players. If the state of system coincides with the origin, then we say that pursuit is completed. In the game, pursuer tries to complete the game, while the aim of evader is opposite. The problem is to find a formula for guaranteed pursuit time. In the present paper, an equation for guaranteed pursuit time is obtained. Moreover, a strategy for the pursuer is constructed in explicit form. To prove the main result, we use solution of a control problem.

Keywords: Differential game, infinite system, pursuer, evader, geometric constraint, control, strategy.

1. Introduction

Differential games, first considered in the book of Isaacs (1965). Since then lots of works with various approaches have been published in developing the theory of differential games for ordinary differential equations.

Control and differential games described by partial differential equations are increasing interest (see, for example, Avdonin and Ivanov (1995), Butkovsky (1969), Chernous' ko (1992), David (1978), Zuazua (2006), Ibragimov (759), Ilin (2001), Lions (1968), Osipov (1975), Satimov and Tukhtasinov (2007), Satimov and Tukhtasinov (2006), Satimov and Tukhtasinov (2005), Tukhtasinov (1995) Tukhtasinov and Mamatov (2008)).

Using decomposition method, some of the control and differential game problems described by parabolic and hyperbolic partial differential equations can be reduced to the ones described by infinite system of ordinary differential equations of first and second order respectively. For example, using this method, one can reduce some control problems for parabolic equation (see, e.g. Avdonin and Ivanov (1995), Butkovsky (1969), Chernous' ko (1992), Ibragimov (759), Satimov and Tukhtasinov (2006), Satimov and Tukhtasinov (2005), Tukhtasinov (1995), Tukhtasinov and Mamatov (2008))

$$\frac{\partial z}{\partial t} + Az = w$$

to the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = w_k, \quad k = 1, 2, \dots, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \to \infty,$$

where w_k, $k = 1, 2, \dots$ are control parameters $z_k, w_k \in R^1$, and λ_k are eigenvalues of the elliptic operator

$$A = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

Because of this significant relationship between control problems described by partial differential equations and those described by infinite system of differential equations, the latter can be investigated within one theoretical framework assuming that the coefficients are any real numbers. This opens up new prospects for the differential games in Hilbert spaces.

The works Ibragimov et al. (2015), Ibragimov et al. (2014), Ibragimov (2013a), Ibragimov (2013b), Ibragimov and Hasim (2010), Ibragimov (2005), Alias et al. (2016), Salimi et al. (2016) relate to such differential games, where various differential game problems for infinite system of differential equations were examined.

In the present paper, we study a pursuit differential game with geometric constraints on the control functions of players in the Hilbert space l_2 with elements

$$\xi = (\xi_1, \xi_2, ..., \xi_k, ...), \quad \sum_{k=1}^{\infty} \xi_k^2 < \infty,$$

where inner product and norm are defined by formulas

$$(\xi, \eta) = \sum_{k=1}^{\infty} \xi_k \eta_k, \quad ||\xi|| = (\xi, \xi)^{1/2}.$$

Differential game is described by the following infinite system of differential equations

$$\begin{array}{ll} \dot{x}_k = -\alpha_k x_k - \beta_k y_k + u_{k1} - v_{k1}, & x_k(0) = x_{k0}, \\ \dot{y}_k = \beta_k x_k - \alpha_k y_k + u_{k2} - v_{k2}, & y_k(0) = y_{k0}, \end{array}, \; k = 1, 2, \dots, \tag{1}$$

where α_k, β_k are real numbers, $\alpha_k \geq 0$,

$$x_0 = (x_{10}, x_{20}, \dots) \in l_2, \; y_0 = (y_{10}, y_{20}, \dots) \in l_2,$$

$u = (u_{11}, u_{12}, u_{21}, u_{22}, \dots)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \dots)$ are control parameters of pursuer and evader, respectively. The aim of the pursuer is to force the state of the system towards the origin of the space l_2 against any action of the evader and the aim of the evader is opposite. We obtain an equation for a guaranteed pursuit time and construct a pursuit strategy explicitly.

Note that a differential game problem was studied for the system (1) in Ibragimov (2013b) when controls of players are subjected to integral constraints. In the present paper, we consider the case of geometric constraints.

2. Statement of Problem

Let ρ_0, ρ and σ be positive numbers ($\rho > \sigma$).

Definition 2.1. *A function*

$$w(\cdot) = (w_{11}(\cdot), w_{12}(\cdot), w_{21}(\cdot), w_{22}(\cdot), \dots), \quad w : [0, T] \to l_2,$$

with measurable components $w_{k1}(t), w_{k2}(t)$, $0 \le t \le T$, $k = 1, 2, \dots$, *subject to the condition*

$$\sum_{k=1}^{\infty}(w_{k1}^2(s) + w_{k2}^2(s)) \le \rho_0^2$$

is referred to as the admissible control, where $T > 0$ *is sufficiently large fixed number. Denote the set of all admissible controls by* $S(\rho_0)$.

Definition 2.2. *Functions* $u(\cdot) \in S(\rho)$ *and* $v(\cdot) \in S(\sigma)$ *are referred to as the admissible controls of pursuer and evader, respectively.*

Definition 2.3. *A function*

$$u(t, v) = (u_1(t, v), u_2(t, v), \dots), \quad u : [0, T] \times l_2 \to l_2,$$

with 2-dimensional components $u_k = (u_{k1}, u_{k2})$ *of the form* $u_k(t, v) = v_k(t) + w_k(t)$, $w_k = (w_{k1}, w_{k2})$, $v_k = (v_{k1}, v_{k2})$, $w(\cdot) = (w_1(\cdot), w_2(\cdot), \dots) \in S(\rho - \sigma)$, *subject to the condition (admissibility)*

$$\sum_{k=1}^{\infty} |u_k(t, v(t))|^2 \le \rho^2 \quad \textit{for any} \quad v(\cdot) \in S(\sigma),$$

is called strategy of pursuer.

Definition 2.4. *If there exists a strategy of the pursuer such that, for any admissible control of the evader, the equality* $z(\tau) = 0$ *occurs at some* τ, $0 \le \tau \le \theta$, *then we say that differential game (1) can be completed for the time* θ. *The time* θ *is called a guaranteed pursuit time.*

Problem. Find a guaranteed pursuit time in differential game (1).

Let

$$z(t) = (z_1(t), z_2(t), \dots) = (x_1(t), y_1(t), x_2(t), y_2(t), \dots),$$

$$z_k(t) = (x_k(t), y_k(t)), \ |z_k| = \sqrt{x_k^2 + y_k^2}, \ \|z\| = \left(\sum_{k=1}^{\infty}(x_k^2 + y_k^2)\right)^{1/2},$$

$$z_0 = (z_{10}, z_{20}, \dots) = (x_{10}, y_{10}, x_{20}, y_{20}, \dots), \ \|z_0\| = \left(\sum_{k=1}^{\infty}(x_{k0}^2 + y_{k0}^2)\right)^{1/2},$$

$$A_k(t) = \begin{bmatrix} e^{-\alpha_k t} cos\beta_k t & -e^{-\alpha_k t} sin\beta_k t \\ e^{-\alpha_k t} sin\beta_k t & e^{-\alpha_k t} cos\beta_k t \end{bmatrix}, \ k = 1, 2, \dots. \tag{2}$$

It is not difficult to show that the matrices $A_k(t)$ have the following properties:

$$\begin{aligned} &A_k(t+h) = A_k(t)A_k(h) = A_k(h)A_k(t), \\ &|A_k(t)z_k| = |A_k^*(t)z_k| = e^{-\alpha_k(t)}|z_k|, \end{aligned} \tag{3}$$

where A^* denotes the transpose of the matrix A.

3. Control Problem

In this section, we construct a control and find a time required to steer the state of a system from a given initial point z_0 to the origin.

Let $C(0,T;l_2)$ be the space of continuous functions $z(\cdot)$ such that $z : [0,T] \to l_2$. We need the following proposition Ibragimov et al. (2008).

Proposition 3.1. *If $w(\cdot) \in S(\rho_0)$ and $\alpha_k \geq 0$, then for any given $T > 0$ the following infinite system of differential equations*

$$\begin{array}{ll} \dot{x}_k = -\alpha_k x_k - \beta_k y_k + w_{1k}, & x_k(0) = x_{k0}, \\ \dot{y}_k = \beta_k x_k - \alpha_k y_k + w_{2k}, & y_k(0) = y_{k0}, \end{array} \quad , \ k = 1, 2, \ldots, \tag{4}$$

has a unique solution $z(t) = (z_1(t), z_2(t), \ldots)$, $0 \leq t \leq T$, in the space $C(0,T;l_2)$. Of course,

$$z_k(t) = A_k(t)z_{k0} + \int_0^t A_k(t-s)w_k(s)ds, \quad k = 1, 2, \ldots.$$

It should be noted that this existence-uniqueness theorem for the system (4) was proved for any finite interval $[0,T]$ (see Ibragimov et al. (2008)).

Before studying differential game for system (1), first we consider the following control problem for the system (4): Find an admissible control $w_{k0}(t)$ and time θ such that $z(\theta) = 0$.

Let

$$\varphi_k(t) = \begin{cases} \frac{e^{2\alpha_k t} - 1}{2\alpha_k}, & \alpha_k > 0, \\ t, & \alpha_k = 0, \end{cases} \quad , \ t > 0.$$

Consider the equation

$$\sum_{k=1}^{\infty} \frac{e^{2\alpha_k t}|z_{k0}|^2}{\varphi_k^2(t)} = \rho_0^2. \tag{5}$$

Since $e^{2\alpha_k t}/\varphi_k^2(t) \to +\infty$ as $t \to 0^+$ for each k, therefore left-hand side of the equation (5) approaches $+\infty$ as $t \to 0^+$. Because of

$$\frac{e^{2\alpha_k t}}{\varphi_k^2(t)} = \frac{1}{t^2}\left[\frac{\alpha_k t}{\sinh(\alpha_k t)}\right]^2 < \frac{1}{t^2},\ \alpha_k > 0,\ \text{and}\ \frac{e^{2\alpha_k t}}{\varphi_k^2(t)} = \frac{1}{t^2},\quad \alpha_k = 0,$$

we obtain

$$\sum_{k=1}^{\infty} \frac{e^{2\alpha_k t}|z_{k0}|^2}{\varphi_k^2(t)} \le \frac{1}{t^2}\sum_{k=1}^{\infty}|z_{k0}|^2 = \frac{1}{t^2}\|z_0\|^2 \to 0,\quad t \to \infty.$$

Moreover, the function $f(t) = \frac{e^{2\alpha_k t}}{\varphi_k^2(t)}$ decreases on $(0,\infty)$ and hence, equation (5) has a unique root $t = \theta > 0$.

Lemma 3.1. *The control*

$$w_{k0}(t) = -\frac{1}{\varphi_k(\theta)}A_k^*(-t)z_{k0},\ 0 \le t \le \theta, \tag{6}$$

steers the system (4) to the origin at the time θ.

Proof. **A.** Show that the control $w_{k0}(t)$ is admissible. Using (3) and the fact that $t = \theta$ is a root of the equation (5), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}|w_k(t)|^2 &= \sum_{k=1}^{\infty}\frac{1}{\varphi_k^2(\theta)}|A_k^*(-t)z_{k0}|^2\\ &= \sum_{k=1}^{\infty}\frac{|z_{k0}|^2}{\varphi_k^2(\theta)}e^{2\alpha_k t}\\ &\le \sum_{k=1}^{\infty}\frac{e^{2\alpha_k\theta}|z_{k0}|^2}{\varphi_k^2(\theta)} = \rho_0^2.\end{aligned}$$

Therefore the control $w_{k0}(t)$ is admissible.

B. Next, show that for the state of system (4) the equation $z(\theta) = 0$ is satisfied. Indeed,

$$\begin{aligned}\xi_k(\theta) &\doteq z_{k0} + \int_0^{\theta} A_k(-s)w_k(s)ds\\ &= z_{k0} - \frac{1}{\varphi_k(\theta)}\int_0^{\theta} A_k(-s)A_k^*(-s)z_{k0}ds\\ &= z_{k0} - z_{k0} = 0,\quad k = 1,2,\dots.\end{aligned} \tag{7}$$

Therefore $z_k(\theta) = A_k(\theta)\xi_k(\theta) = 0,\ k = 1, 2, \ldots$. This completes the proof of the lemma. $\square$

4. Differential Game Problem

We now turn to the system (1) and formulate the main result of the paper. Let $t = \theta_1$ be a root of the equation

$$\sum_{k=1}^{\infty} \frac{e^{2\alpha_k t}|z_{k0}|^2}{\varphi_k^2(t)} = (\rho - \sigma)^2. \tag{8}$$

As mentioned above, this root is unique.

Theorem 4.1. *The number θ_1 is a guaranteed pursuit time in the game (1).*

Proof. First, show that pursuit is completed at the time θ_1. Let the pursuer use the following strategy:

$$u_k(t, v_k(t)) = v_k(t) + \omega_{k0}(t), \quad \omega_{k0}(t) = -\frac{1}{\varphi_k(\theta_1)} A_k^*(-t) z_{k0}, \quad k = 1, 2, \ldots, \tag{9}$$

Pursuit is completed at the time θ_1 since using (9) and (8) we obtain

$$\begin{aligned}
z_k(\theta_1) &= A_k(\theta_1) z_{k0} + \int_0^{\theta_1} A_k(\theta_1 - s)\left(u_k(s, v_k(s)) - v_k(s)\right) ds \\
&= A_k(\theta_1) z_{k0} + \int_0^{\theta_1} A_k(\theta_1 - s)\omega_{k0}(s) ds \\
&= A_k(\theta_1)\left[z_{k0} - \frac{1}{\varphi_k(\theta_1)} \int_0^{\theta_1} A_k(-s) A_k^*(-s) z_{k0} ds\right] = 0.
\end{aligned}$$

Thus, $z_k(\theta_1) = 0, \quad k = 1, 2, \ldots$

Next, show admissibility of strategy (9). Using the Minkowskii inequality and (9) yields

$$
\begin{aligned}
\left(\sum_{k=1}^{\infty} u_k^2(t)\right)^{1/2} &= \left(\sum_{k=1}^{\infty} |v_k(t) + \omega_{k0}(t)|^2\right)^{1/2} \\
&\le \left(\sum_{k=1}^{\infty} (|v_k(t)| + |\omega_{k0}(t)|)^2\right)^{1/2} \\
&\le \left(\sum_{k=1}^{\infty} |v_k(t)|^2\right)^{1/2} + \left(\sum_{k=1}^{\infty} \frac{e^{2\alpha_k\theta_1}}{\varphi_k^2(\theta_1)} |z_{k0}|^2\right)^{1/2} \\
&\le \sigma + \rho - \sigma = \rho.
\end{aligned}
$$

This completes the proof of the theorem. □

5. Conclusion

We have studied a pursuit differential game described by an infinite system of first-order differential equations when control functions are subjected to integral constraints. First, we solved an auxiliary control problem: we have constructed a control function and found a time for which state of the system can be steered to the origin. We have then solved pursuit problem: constructed a pursuit strategy, and found guaranteed pursuit time.

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