.

UNIVERSITI PUTRA MALAYSIA

PROVING KOCHEN-SPECKER THEOREM USING PROJECTION MEASUREMENT AND POSITIVE OPERATOR-VALUED MEASURE

TOH SING POH

IPM 2008 3

PROVING KOCHEN-SPECKER THEOREM USING PROJECTION MEASUREMENT AND POSITIVE OPERATOR-VALUED MEASURE

TOH SING POH

DOCTOR OF PHILOSOPHY
UNIVERSITI PUTRA MALAYSIA

2008

PROVING KOCHEN-SPECKER THEOREM USING PROJECTION MEASUREMENT AND POSITIVE OPERATOR-VALUED MEASURE

By

TOH SING POH

Thesis Submitted to the School of Graduate Studies, Unversiti Putra Malaysia, in fulfilment of the Requirements for the Doctor of Philosophy

December 2008

To my mum, who never formally educated but has always shown support on my pursuit of knowledge. She passed away before my completion of study.

To my wife, who has always respected my research life.

Abstract of the thesis presented to the Senate of Universiti Putra Malaysia
in fulfilment of requirement for the Doctor of Philosophy

PROVING KOCHEN-SPECKER THEOREM USING PROJECTION MEASUREMENT AND POSITIVE OPERATOR-VALUED MEASURE

By

TOH SING POH

December 2008

Chairman : Associate Professor Hishamuddin Zainuddin, PhD

Faculty : Institute for Mathematical Research

One of the main theorems on the impossibility of hidden variables in quantum mechanics is Kochen-Specker theorem (KS). This theorem says that any hidden variable theory that satisfies quantum mechanics must be contextual. More specifically, it asserts that, in Hilbert space of dimension ≥ 3, it is impossible to associate definite numerical values, 1 or 0, with every projection operator P_m, in such a way that, if a set of commuting P_m satisfies $\sum P_m = 1$, the corresponding values $v(P_m)$ will also satisfy $\sum v(P_m) = 1$. Since the first proof of Kochen and Specker using 117 vectors in R^3, there were many attempts to reduce the number of vector either via conceiving ingenious models or extending the system being considered to higher dimension. By considering eight dimensional three qubits system, we found a state dependent proof that requires only five vectors. The state that we assign value of 1 is the ray that arises from intersection of two planes.

The recent advancements show that the KS theorem proof can be extended to two dimensional quantum system through generalized measurement represented by positive operator-valued measured (POVM). In POVMs the number of available outcomes of a measurement may be higher than the dimensionality of the Hilbert space and N-outcome generalized measurement is represented by N-element POVM which consists of N positive semidefinite operators $\{E_d\}$ that sum to identity. Each pair of elements is not mutually orthogonal if the number of outcome of measurements is bigger than the dimensionality. In terms of POVM, Kochen-Specker theorem asserts that $\sum_i E_i = I$ and $\sum_i v(E_i) = I$ could not be satisfied for $d \geq 2$. We developed a general model that enables us to generate different sizes of the POVM for the proof of the Kochen-Specker theorem. We show that the current simplest Nakamura model is in fact a special case of our model. W also provide another model which is as simple as the Nakamura's but consists of different sets of POVM.

Abstrak thesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Doktor Falsafah.

PEMBUKTIAN TEOREM KOCHEN-SPECKER DENGAN MENGGUNAKAN PENGUKURAN OPERATOR UNJURAN DAN PENGUKURAN OPERATOR-NILAI POSITIF

Oleh

TOH SING POH

Disember 2008

Pengerusi : Profesor Madya Hishamuddin Zainuddin, PhD

Fakulti : Institut Penyelidikan Matematik

Salah satu teorem utama mengenai ketidakwujudan pembolehubah tersembunyi dalam mekanik kuantum adalah teorem Kochen-Specker (KS). Teorem in menyatakan bahawa sebarang teori pengbolehubah tersembunyi yang mematuhi mekanik kuantum mestilah kontekstual. Secara spesifik, ia menyatakan bahawa di dalam ruang Hilbert yang berdimensi ≥ 3, adalah mustahil untuk memberi nilai tentu, samada 1 atau 0, kepada operator unjuran P_m di mana jika satu set P_m bertukar tertib memenuhi $\sum P_m = 1$, nilai-nilai padanan $v(P_m)$ juga memenuhi $\sum v(P_m) = 1$. Sejak pembuktian pertama yang dikemukakan oleh Kochen dan Specker yang menggunakan 117 vektors dalam R^3, terdapat banyak usaha untuk mengurangkan bilangan vektor samada dengan mengemukakan model yang bijak atau mempertimbangkan sistem yang berdimensi lebih tinggi. Dengan mempertimbangkan sistem tiga qubit lapan dimensi, kami mengemukakan pembuktian yang bergantung kepada keadaan, memerlukan hanya lima vektor. Keadaan yang diberikan nilai 1 adalah hasilan daripada persilangan dua muka.

Perkembangaan baru-baru ini menunjukkan bahawa pembuktian teorem Kochen- Specker dapat diperluaskan kepada sistem kuantum berdimensi dua melalui pengukuran teritlak yang diwakili oleh ukuran bernilai operator positif (UBOP). Dalam UBOP bilangan dapatan pengukuran adalah berkemungkinan lebih daripada dimensi ruang Hilbert dan pengukuran teritlak *N*-dapatan diwakili oleh UBOP *N*-unsur yang terdiri daripada *N* operator separa-tentu positif $\{E_i\}$ yang terjumlah ke identiti. Setiap pasangan unsur tidak saling ortogon jika bilangan hasilan pengukuran adalah lebih daripada dimensi ruang. Dalam bentuk UBOP, teorem Kochen-Specker menyatakan bahawa $\sum_i E_i = I$ dan $\sum_i v(E_i) = I$ tidak dapat dipatuhi bagi $d \geq 2$. Kami mengemukakan satu model umum yang membolehkan kami mengitlakkan saiz UBOP yang berbeza dalam pembuktian teorem Kochen-Specker. Kami menunjukkan bahawa model Nakamura yang termudah kini sebenarnya adalah kes khas dalam model kami. Kami juga membekalkan satu model lain yang seringkas model Nakamura tetapi terdiri dariapda set UBOP yang berlainan.

ACKNOWLEDGEMENTS

There are so many interesting topics in physics. Ironically, the broadness of my interest in certain sense signify the uncertain direction that I need to focus on and it in turns becomes one of the sources of depression in my academic life, especially whenever I somehow sense that the progress is too slow. I would like to thank my supervisor, Assoc. Prof. Dr. Hishamuddin, for playing his key role of leading me to work on fundamental problems of quantum mechanics and arouse my willingness to devote myself in this area for my future.

I would also like to thank my co-supervisors, Assoc. Prof. Dr. Jumiah Hassan and Dr. Isamiddin S. Rakhimov for their willingness to help and spend their invaluable time in examining my thesis.

I would like to thank Mr. Foo Kim Eng who is my good consultant in MatLab.

I certify that a Thesis Examination Committee has met on 4 December 2008 to conduct the final examination of Toh Sing Poh on his thesis entitled "Proving Kochen-Specker Theorem Using Projection Measurement and Positive Operator-Valued Measure" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

Members of the Examination Committee were as follows:

Mohanad Rushdan Md. Said, PhD
Associate Professor
Faculty for Mathematical Research,
Universiti Putra Malaysia.
(Chairman)

Ionel Valeriu Grozescu, PhD
Associate Professor
Faculty of Science,
Universiti Putra Malaysia.
(Internal Examiner)

Zuriati Ahmad Zukarnain, PhD
Lecturer
Faculty of Computer Science and Information Technology,
Universiti Putra Malaysia
(Internal Examiner)

Kwek Leong Chuan, PhD
Associate Professor
National Institute of Education and Center for Quantum Technologies,
National University of Singapore
(External Examiner)

HASANAH MOHD. GHAZALI, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 29 January 2009

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

Hishamuddin Zainuddin, PhD
Associate Professor
Institute for Mathematical Research,
Universiti Putra Malaysia.
(Chairman)

Jumiah Hassan, PhD
Associate Professor
Faculty of Science and Environmental Studies,
Universiti Putra Malaysia.
(Member)

Isamiddin S. Rakhimov, PhD
Institute for Mathematical Research,
Universiti Putra Malaysia.
(Member)

HASANAH MOHD. GHAZALI, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 12-2-2009

DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

Toh Sing Poh

Date:

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

$\lvert\varphi\rangle$	State
$\lvert e_i\rangle$	i^{th} basis vector
$\otimes$	Tensor product
R_i	Ray labeled as i
P_i	Projection operators corresponding to R_i
$\hat{\Omega}$	Operator
$\hat{\rho}$	Density Operator
$\alpha, \beta, \ldots$	Complex number
$V^n(F)$	n-dimensional vector space over a field F
$\mathcal{H}$	Hilbert space
O	Observable
Tr	Trace
$X^{\dagger}$	Transpose conjugate of X
σ_i	Pauli matrice
λ_i	i^{th} Eigenvalues
$p(\lambda_i)$	Probability to get result as λ_i
$E(O)$	Expectation value of measurement upon observable O
$\hat{E}_i$	i^{th} positive semidefinite operator

$\hat{K}$	Krauss Operator
HVT	Hidden Variable Theory
POVM	Positive Operator-Valued Measure

CHAPTER 1

INTRODUCTION

1. Introduction

Quantum theory has been born for over 100 years, but there are quite a number of fundamental questions that still puzzle physicists to date. Quantum weirdness that have been widely attracting physicists are nonlocality, contextuality, entanglement, indeterminancy, etc. The relations between idea of hidden variables and nonlocality are deeply studied in Bell theorem, whereas relations between hidden variables and contextuality are studied in Kochen-Specker (KS) theorem. Our study focuses on the latter which states that a hidden variable theory must be contextual. The first proof of the KS theorem was provided by Bell in 1966 and independently by Kochen and Specker in 1967 [1].

1.1 Contextuality

In order to reveal the properties of a physical system, an observer needs to do measurement on the system. Measurement generates result that implies whether a certain property is associated with physical system. For example, to check whether the color of a car is red, our eyes need to receive light reflected from the car. This is in fact a measurement process. Our perception about the color is associated with the result of measurement that tells us whether the car is red. For simplicity, we would represent a positive result (existence of particular property) as 1, whereas representing a negative result (non-existence of particular property) is 0.

Suppose we measure both color and weight of the car at the same time. We would say that measurement of color and weight are in the same context of measurement. It is definitely possible to measure the color in different measurement context, for instance we often get to know about the color and speed of a car at the same time. In this case, measurement of color and speed would form another context. In classical physics, the results of measurement are independent of contexts. All of us believe that our perception of color would not be different so as to depend on whether we measure it together with weight or speed.

Through out the thesis we refer measurement to the determination of intrinsic angular momentum or spin of the quantum system being considered. Spin is one of the properties of quantum system that does not have a counterpart in classical system. It is a nonclassical degree of freedom characterized by a number which can take values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Some elementary particles when passed through a nonuniform magnetic field they would be deflected either up or down; some would be deflected either up, down or non-deflection. The deflection is due to the interaction between spin and magnetic field. The interaction is a measurement that reveals the spin of the particle. Particles with only two possible exclusive deflections are called spin-$\frac{1}{2}$ particles, whereas particles with three possible exclusive deflections are called spin-1 particles.