



**UNIVERSITI PUTRA MALAYSIA**

**PROVING KOCHEN-SPECKER THEOREM USING PROJECTION MEASUREMENT  
AND POSITIVE OPERATOR-VALUED MEASURE**

**TOH SING POH**

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**DOCTOR OF PHILOSOPHY  
UNIVERSITI PUTRA MALAYSIA**

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USING PROJECTION MEASUREMENT AND  
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**By**

**TOH SING POH**

**Thesis Submitted to the School of Graduate Studies, Unversiti Putra Malaysia,  
in fulfilment of the Requirements for the Doctor of Philosophy**

**December 2008**



To my mum, who never formally educated but has always shown support on my pursuit of knowledge. She passed away before my completion of study.

To my wife, who has always respected my research life.



Abstract of the thesis presented to the Senate of Universiti Putra Malaysia  
in fulfilment of requirement for the Doctor of Philosophy

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One of the main theorems on the impossibility of hidden variables in quantum mechanics is Kochen-Specker theorem (KS). This theorem says that any hidden variable theory that satisfies quantum mechanics must be contextual. More specifically, it asserts that, in Hilbert space of dimension  $\geq 3$ , it is impossible to associate definite numerical values, 1 or 0, with every projection operator  $P_m$ , in such a way that, if a set of commuting  $P_m$  satisfies  $\sum P_m = 1$ , the corresponding values  $v(P_m)$  will also satisfy  $\sum v(P_m) = 1$ . Since the first proof of Kochen and Specker using 117 vectors in  $\mathbb{R}^3$ , there were many attempts to reduce the number of vector either via conceiving ingenious models or extending the system being considered to higher dimension. By considering eight dimensional three qubits system, we found a state dependent proof that requires only five vectors. The state that we assign value of 1 is the ray that arises from intersection of two planes.



The recent advancements show that the KS theorem proof can be extended to two dimensional quantum system through generalized measurement represented by positive operator-valued measured (POVM). In POVMs the number of available outcomes of a measurement may be higher than the dimensionality of the Hilbert space and N-outcome generalized measurement is represented by N-element POVM which consists of N positive semidefinite operators  $\{E_d\}$  that sum to identity. Each pair of elements is not mutually orthogonal if the number of outcome of measurements is bigger than the dimensionality. In terms of POVM, Kochen-Specker theorem asserts that  $\sum_i E_i = I$  and  $\sum_i v(E_i) = I$  could not be satisfied for  $d \geq 2$ . We developed a general model that enables us to generate different sizes of the POVM for the proof of the Kochen-Specker theorem. We show that the current simplest Nakamura model is in fact a special case of our model. We also provide another model which is as simple as the Nakamura's but consists of different sets of POVM.



Abstrak thesis yang dikemukakan kepada Senat Universiti Putra Malaysia  
sebagai memenuhi keperluan untuk Ijazah Doktor Falsafah.

**PEMBUKTIAN TEOREM KOCHEN-SPECKER  
DENGAN MENGGUNAKAN PENGUKURAN OPERATOR UNJURAN DAN  
PENGUKURAN OPERATOR-NILAI POSITIF**

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Salah satu teorem utama mengenai ketidakwujudan pembolehubah tersembunyi dalam mekanik kuantum adalah teorem Kochen-Specker (KS). Teorem ini menyatakan bahawa sebarang teori pembolehubah tersembunyi yang mematuhi mekanik kuantum mestilah kontekstual. Secara spesifik, ia menyatakan bahawa di dalam ruang Hilbert yang berdimensi  $\geq 3$ , adalah mustahil untuk memberi nilai tentu, samada 1 atau 0, kepada operator unjuran  $P_m$  di mana jika satu set  $P_m$  bertukar tertib memenuhi  $\sum P_m = 1$ , nilai-nilai padanan  $v(P_m)$  juga memenuhi  $\sum v(P_m) = 1$ . Sejak pembuktian pertama yang dikemukakan oleh Kochen dan Specker yang menggunakan 117 vektors dalam  $R^3$ , terdapat banyak usaha untuk mengurangkan bilangan vektor samada dengan mengemukakan model yang bijak atau mempertimbangkan sistem yang berdimensi lebih tinggi. Dengan mempertimbangkan sistem tiga qubit lapan dimensi, kami mengemukakan pembuktian yang bergantung kepada keadaan, memerlukan hanya lima vektor. Keadaan yang diberikan nilai 1 adalah hasilan daripada persilangan dua muka.



Perkembangan baru-baru ini menunjukkan bahawa pembuktian teorem Kochen- Specker dapat diperluaskan kepada sistem kuantum berdimensi dua melalui pengukuran teritlak yang diwakili oleh ukuran bernilai operator positif (UBOP). Dalam UBOP bilangan dapatan pengukuran adalah berkemungkinan lebih daripada dimensi ruang Hilbert dan pengukuran teritlak  $N$ -dapatan diwakili oleh UBOP  $N$ -unsur yang terdiri daripada  $N$  operator separa-tentu positif  $\{E_i\}$  yang terjumlah ke identiti. Setiap pasangan unsur tidak saling ortogon jika bilangan hasilan pengukuran adalah lebih daripada dimensi ruang. Dalam bentuk UBOP, teorem Kochen-Specker menyatakan bahawa  $\sum_i E_i = I$  dan  $\sum_i v(E_i) = I$  tidak dapat dipatuhi bagi  $d \geq 2$ . Kami mengemukakan satu model umum yang membolehkan kami mengitlakkan saiz UBOP yang berbeza dalam pembuktian teorem Kochen-Specker. Kami menunjukkan bahawa model Nakamura yang termudah kini sebenarnya adalah kes khas dalam model kami. Kami juga membekalkan satu model lain yang ringkas model Nakamura tetapi terdiri daripada set UBOP yang berlainan.



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I certify that a Thesis Examination Committee has met on 4 December 2008 to conduct the final examination of Toh Sing Poh on his thesis entitled “Proving Kochen-Specker Theorem Using Projection Measurement and Positive Operator-Valued Measure” in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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## DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

---

Toh Sing Poh

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## LIST OF SYMBOLS AND ABBREVIATIONS

$ \varphi\rangle$	State
$ e_i\rangle$	$i^{\text{th}}$ basis vector
$\otimes$	Tensor product
$R_i$	Ray labeled as $i$
$P_i$	Projection operators corresponding to $R_i$
$\hat{\Omega}$	Operator
$\hat{\rho}$	Density Operator
$\alpha, \beta, \dots$	Complex number
$V^n(F)$	$n$ -dimensional vector space over a field $F$
$\mathcal{H}$	Hilbert space
$O$	Observable
$Tr$	Trace
$X^\dagger$	Transpose conjugate of $X$
$\sigma_i$	Pauli matrice
$\lambda_i$	$i^{\text{th}}$ Eigenvalues
$p(\lambda_i)$	Probability to get result as $\lambda_i$
$E(O)$	Expectation value of measurement upon observable $O$
$\hat{E}_i$	$i^{\text{th}}$ positive semidefinite operator

$\hat{K}$	Krauss Operator
HVT	Hidden Variable Theory
POVM	Positive Operator-Valued Measure



# CHAPTER 1

## INTRODUCTION

### 1. Introduction

Quantum theory has been born for over 100 years, but there are quite a number of fundamental questions that still puzzle physicists to date. Quantum weirdness that have been widely attracting physicists are nonlocality, contextuality, entanglement, indeterminacy, etc. The relations between idea of hidden variables and nonlocality are deeply studied in Bell theorem, whereas relations between hidden variables and contextuality are studied in Kochen-Specker (KS) theorem. Our study focuses on the latter which states that a hidden variable theory must be contextual. The first proof of the KS theorem was provided by Bell in 1966 and independently by Kochen and Specker in 1967 [1].

#### 1.1 Contextuality

In order to reveal the properties of a physical system, an observer needs to do measurement on the system. Measurement generates result that implies whether a certain property is associated with physical system. For example, to check whether the color of a car is red, our eyes need to receive light reflected from the car. This is in fact a measurement process. Our perception about the color is associated with the result of measurement that tells us whether the car is red. For simplicity, we would represent a





positive result (existence of particular property) as 1, whereas representing a negative result (non-existence of particular property) is 0.

Suppose we measure both color and weight of the car at the same time. We would say that measurement of color and weight are in the same context of measurement. It is definitely possible to measure the color in different measurement context, for instance we often get to know about the color and speed of a car at the same time. In this case, measurement of color and speed would form another context. In classical physics, the results of measurement are independent of contexts. All of us believe that our perception of color would not be different so as to depend on whether we measure it together with weight or speed.

Through out the thesis we refer measurement to the determination of intrinsic angular momentum or spin of the quantum system being considered. Spin is one of the properties of quantum system that does not have a counterpart in classical system. It is a nonclassical degree of freedom characterized by a number which can take values  $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ . Some elementary particles when passed through a nonuniform magnetic field they would be deflected either up or down; some would be deflected either up, down or non-deflection. The deflection is due to the interaction between spin and magnetic field. The interaction is a measurement that reveals the spin of the particle. Particles with only two possible exclusive deflections are called spin- $\frac{1}{2}$  particles, whereas particles with three possible exclusive deflections are called spin-1 particles.