UNIVERSITI PUTRA MALAYSIA

GENERALIZATIONS OF LINDELÖF PROPERTIES IN BITOPOLOGICAL SPACES

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATION

To my parents, father and mother in-law

and

to my wife and my kids

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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October 2007

Chairman: Professor Adem Kılıçman, PhD

Institute : Mathematical Research

A bitopological space (X, τ_1, τ_2) is a set X together with two (arbitrary) topologies τ_1 and τ_2 defined on X. The first significant investigation into bitopological spaces was launched by J. C. Kelly in 1963. He recognized that by relaxing the symmetry condition on pseudo-metrics, two topologies were induced by the resulting quasi-pseudo-metrics. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Some such extension are pairwise regular, pairwise Hausdorff and pairwise normal spaces. There are several works dedicated to the investigation of bitopologies; most of them deal with the theory itself but very few with applications. In this thesis, we are concerned with the ideas of pairwise Lindelöfness, generalizations of pairwise Lindelöfness and generalizations of pairwise regular-Lindelöfness in bitopological spaces motivated by the known ideas of Lindelöfness, generalized Lindelöfness and generalized regular-Lindelöfness in topological spaces.

There are four kinds of pairwise Lindelöf space namely Lindelöf, *B*-Lindelöf, *s*-Lindelöf and *p*-Lindelöf spaces that depend on open, *i*-open, $\tau_1\tau_2$-open and *p*-open

covers respectively introduced by Reilly in 1973, and Fora and Hdeib in 1983. For instance, a bitopological space X is said to be p-Lindelöf if every p-open cover of X has a countable subcover. There are three kinds of generalized pairwise Lindelöf space namely pairwise nearly Lindelöf, pairwise almost Lindelöf and pairwise weakly Lindelöf spaces that depend on open covers and pairwise regular open covers. Another idea is to generalize pairwise regular-Lindelöfness to bitopological spaces. This leads to the classes of pairwise nearly regular-Lindelöf, pairwise almost regular-Lindelöf and pairwise weakly regular-Lindelöf spaces that depend on pairwise regular covers.

Some characterizations of these generalized Lindelöf bitopological spaces are given. The relations among them are studied and some counterexamples are given in order to prove that the generalizations studied are proper generalizations of Lindelöf bitopological spaces. Subspaces and subsets of these spaces are also studied, and some of their characterizations investigated. We show that some subsets of these spaces inherit these generalized pairwise covering properties and some others, do not.

Mappings and generalized pairwise continuity are also studied in relation to these generalized pairwise covering properties and we prove that these properties are bitopological properties. Some decompositions of pairwise continuity are defined and their properties are studied. Several counterexamples are also given to establish the relations among these generalized pairwise continuities. The effect of mappings, some decompositions of pairwise continuity and some generalized pairwise openness mappings on these generalized pairwise covering properties are investigated. We show that some proper mappings preserve these pairwise covering properties such as: pairwise δ-continuity preserves the pairwise nearly Lindelöf property; pairwise θ-continuity preserves the pairwise almost Lindelöf property; pairwise almost continuity preserves the pairwise weakly Lindelöf, pairwise almost regular-Lindelöf and pairwise weakly regular-Lindelöf properties; and pairwise R-maps preserve the pairwise nearly regular-Lindelöf property. Moreover, we give some conditions on the maps or on the spaces which ensure that weak forms of pairwise continuity preserve some of these generalized pairwise covering properties.

Furthermore, it is shown that all the generalized pairwise covering properties are satisfy the pairwise semiregular invariant properties where some of them satisfy the pairwise semiregular properties. On the other hand, none of the pairwise Lindelöf properties are pairwise semiregular properties. The productivity of these generalized pairwise covering properties are also studied. It is well known by Tychonoff Product Theorem that compactness and pairwise compactness are preserved under products. We show by means of counterexamples that in general the pairwise Lindelöf, pairwise nearly Lindelöf and similar properties are not even preserved under finite products. We give some necessary conditions, for example the P-space property; under which these generalized pairwise covering properties become finitely productive.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENGITLAKAN SIFAT-SIFAT LINDELÖF DALAM RUANG BITOPOLOGI

Oleh

ZABIDIN BIN SALLEH

Oktober 2007

Pengerusi: Profesor Adem Kılıçman, PhD

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Suatu ruang bitopologi (X, τ_1, τ_2) ialah suatu set X bersama dengan dua (sebarangan) topologi τ_1 dan τ_2 ditakrifkan pada X. Penyelidikan bermakna yang pertama dalam ruang bitopologi telah dimulakan oleh J. C. Kelly dalam tahun 1963. Beliau memperkenalkannya dengan merehatkan syarat simetri pada metrik-pseudo supaya dua topologi diaruh oleh metrik-quasi-pseudo yang terhasil. Selanjutnya, Kelly telah mengitlakkan beberapa hasil piawai bagi aksiom-aksiom pemisahan dalam suatu ruang topologi kepada suatu ruang bitopologi. Beberapa pengitlakan tersebut ialah ruang sekata pasangan demi pasangan, Hausdorff pasangan demi pasangan dan normal pasangan demi pasangan. Terdapat beberapa kerja yang didedikasikan terhadap penyelidikan bitopologi ini; kebanyakannya menangani teori itu sendiri tetapi sangat sedikit aplikasinya. Dalam tesis ini, kita menumpukan kepada idea keLindelöfan pasangan demi pasangan, pengitlakan keLindelöfan pasangan demi pasangan dan pengitlakan keLindelöfan-sekata pasangan demi pasangan dalam ruang bitopologi dimotivasikan oleh idea-idea yang diketahui bagi keLindelöfan, keLindelöfan teritlak dan keLindelöfan-sekata teritlak dalam ruang topologi.

Terdapat empat jenis ruang Lindelöf pasangan demi pasangan iaitu ruang Lindelöf, *B*-Lindelöf, *s*-Lindelöf dan *p*-Lindelöf yang bergantung kepada tudung terbuka, *i*-terbuka, $\tau_1\tau_2$-terbuka dan *p*-terbuka masing-masing diperkenalkan oleh Reilly dalam tahun 1973, dan Fora dan Hdeib dalam tahun 1983. Misalannya, suatu ruang bitopologi X dikatakan *p*-Lindelöf jika setiap tudung *p*-terbuka mempunyai suatu subtudung terbilangkan. Terdapat tiga jenis pengitlakan ruang Lindelöf pasangan demi pasangan, iaitu ruang dekat Lindelöf pasangan demi pasangan, hampir Lindelöf pasangan demi pasangan dan lemah Lindelöf pasangan demi pasangan yang bergantung kepada tudung terbuka dan tudung terbuka sekata pasangan demi pasangan. Idea lain ialah pengitlakan keLindelöfan-sekata pasangan demi pasangan dalam ruang bitopologi. Ini membawa kepada kelas bagi ruang dekat Lindelöf-sekata pasangan demi pasangan, hampir Lindelöf-sekata pasangan demi pasangan dan lemah Lindelöf-sekata pasangan demi pasangan yang bergantung kepada tudung sekata pasangan demi pasangan.

Beberapa ciri-ciri bagi ruang bitopologi Lindelöf teritlak diberikan. Hubungan diantara mereka dikaji dan beberapa contoh penyangkal diberikan untuk membuktikan pengitlakan yang dikaji itu adalah pengitlakan yang wajar bagi ruang bitopologi Lindelöf. Subruang dan subset bagi ruang-ruang ini juga dikaji, dan beberapa ciri-cirinya diselidik. Kita tunjukkan bahawa beberapa subset bagi ruang-ruang ini mewarisi sifat-sifat tudung pasangan demi pasangan teritlak dan beberapa yang lain, tidak.

Pemetaan dan keselanjaran pasangan demi pasangan teritlak juga dikaji pada sifat-sifat tudung pasangan demi pasangan teritlak ini dan kita membuktikan bahawa sifat-

sifat ini adalah sifat-sifat bitopologi. Beberapa penguraian keselanjaran pasangan demi pasangan ditakrifkan dan sifat-sifat mereka dikaji. Pelbagai contoh-contoh penyangkal juga diberikan untuk mengukuhkan hubungan diantara keselanjaran pasangan demi pasangan teritlak ini. Kesan bagi pemetaan, beberapa penguraian keselanjaran pasangan demi pasangan dan beberapa pemetaan terbuka pasangan demi pasangan teritlak pada sifat-sifat tudung pasangan demi pasangan teritlak ini diselidik. Kita tunjukkan bahawa beberapa pemetaan yang wajar, mengekalkan sifat-sifat tudung pasangan demi pasangan teritlak ini seperti: δ-keselanjaran pasangan demi pasangan mengekalkan sifat dekat Lindelöf pasangan demi pasangan; θ-keselanjaran pasangan demi pasangan mengekalkan sifat hampir Lindelöf pasangan demi pasangan; hampir keselanjaran pasangan demi pasangan mengekalkan sifat-sifat lemah Lindelöf pasangan demi pasangan, hampir Lindelöf-sekata pasangan demi pasangan dan lemah Lindelöf-sekata pasangan demi pasangan; dan peta-*R* pasangan demi pasangan mengekalkan sifat dekat Lindelöf-sekata pasangan demi pasangan. Seterusnya, kita memberikan beberapa syarat pada pemetaan ataupun ruang yang memastikan bahawa bentuk-bentuk lemah bagi keselanjaran pasangan demi pasangan mengekalkan beberapa sifat-sifat tudung pasangan demi pasangan teritlak ini.

Selanjutnya, telah ditunjukkan bahawa semua sifat-sifat tudung pasangan demi pasangan teritlak adalah memuaskan sifat-sifat tak varian separa-sekata pasangan demi pasangan dimana sebahagiannya memuaskan sifat-sifat separa-sekata pasangan demi pasangan. Sebaliknya, tiada sifat-sifat Lindelöf pasangan demi pasangan yang memuaskan sifiat-sifat separa-sekata pasangan demi pasangan. Pendaraban bagi sifat-sifat tudung pasangan demi pasangan teritlak ini juga dikaji. Ianya diketahui baik oleh Teorem Hasil Darab Tychonoff bahawa kepadatan dan kepadatan pasangan demi pasangan adalah dikekalkan dibawah hasil darab. Kita tunjukkan dengan contoh-contoh penyangkal bahawa secara amnya Lindelöf pasangan demi pasangan, dekat Lindelöf pasangan demi pasangan dan sifat-sifat serupa dengannya adalah tidak dikekalkan dibawah hasil darab terhingga. Kita berikan beberapa syarat perlu, contohnya sifat ruang-*P*; di bawah yang sifat-sifat tudung pasangan demi pasangan teritlak menjadi pendaraban terhinggakan.

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I certify that an Examination Committee met on 31st October 2007 to conduct the final examination of Zabidin bin Salleh on his Doctor of Philosophy thesis entitled "Generalizations of Lindelöf Properties in Bitopological Spaces" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the degree of Doctor of Philosophy. Members of the Examination Committee were as follows:

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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Date: 21 February 2008

DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

ZABIDIN BIN SALLEH

Date:

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LIST OF ABBREVIATIONS

i-int (A) or i-int$_X$ (A)	interior of A in X with respect to τ_i
i-cl (A) or i-cl$_X$ (A)	closure of A in X with respect to τ_i
$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{Q}$	set of rational numbers
$\aleph_0$	cardinality of $\mathbb{N}$
$\subseteq$	Subset
$X \setminus A$	complement of A in X
(X, τ)	topological space X
(X, τ_1, τ_2)	bitopological space X
$\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$	pairwise semiregularization of (X, τ_1, τ_2)
$\tau\|_A$	induced (or relative) topology τ on A
$(A, \tau_1\|_A, \tau_2\|_A)$	subspace of X whenever $A \subseteq X$
τ_d	discrete topology
τ_{ind}	indiscrete topology
τ_u	usual (Euclidean) topology
τ_{cof}	cofinite (finite complement) topology
τ_{coc}	cocountable (countable complement) topology
τ_s	Sorgenfrey topology

$\tau_{l.r}$	left ray topology
$\tau_{r.r}$	right ray topology
iff	if and only if
Δ	index set

CHAPTER 1

INTRODUCTION

In theoretical and applied areas of mathematics we frequently deal with sets endowed with various structures. However, it may happen that the consideration of a set with a specific structure, say topological, algebraic, order, uniform, convex etcetera is not sufficient to solve the problem posed and in that case, it becomes necessary to introduce an additional structure on the set under consideration. To confirm this idea, it will do to recall the theories of topological groups, ordered topological spaces, topological spaces with measure, convex topological structures and others. This list is not complete without adding the theory of bitopological spaces.

The notion of a bitopological space (X, τ_1, τ_2), that is, a set X equipped with two arbitrary topologies τ_1 and τ_2, was first formulated by J. C. Kelly [29] in 1963. Kelly investigated non-symmetric distance functions, the so-called quasi-pseudo-metrics on $X \times X$, that generate two topologies on X that, in general, are independent of each other. Previously, such non-symmetric distance functions had been studied by Wilson [87]. Distance functions, uniformity and proximity are the related notions in defining the topology and, naturally, the situation treated in [29] is by no means the only way leading to a non-symmetric occurrence of two topologies on the same set; the investigations of quasi-uniformity and quasi-proximity (see [15], [23], [61]) also lead to an analogous results.

From the above said, it follows that due to the specific properties of the considered structures two topologies are frequently generated on the same set and can be either independent of each other though symmetric by construction or closely interconnected. Certainly, the investigation of a set with two topologies, interconnected by relations of "bitopological" character, makes it possible on some occasions to obtain a combined effect, that is, to get more information than we would acquire if we considered the same set with each topology separately.

If we compare all the results available in the theory of bitopological spaces from the general point of view, we shall find that in different cases two topologies on a set are not, generally speaking, interconnected by some common law that takes place for all bitopological spaces. However if, when defining a bitopological notion, the closure and interior operators are successively applied in an arbitrary initial order to the same set, then in general, these operators will interchange in topologies as well. For example, the reader may refer to the concepts of (i, j)-regular open and (i, j)-regular closed sets in Chapter 2.

Bitopological spaces that we are going to discuss is one of the research interest recently in the area of topology. It should be also noted that at present, there are several hundred works dedicated to the investigation of bitopologies; most of them deal with the theory itself but very few deal with applications. Before starting, we recall that, spaces mean topological spaces or bitopological spaces on which no separation axioms nor bitopological separation axioms are assumed unless explicitly stated otherwise, and the bitopological space (X, τ_1, τ_2) will be replaced by X if there is no chance for confusion.

1.1 Historical Remarks

A topological space (X, τ) is said to be Lindelöf if every open cover of X has a countable subcover. The Lindelöf Theorem, that every second countable space is Lindelöf, was proved for Euclidean spaces as early as 1903 by Ernest Leonard Lindelöf (1870-1946) (*Sur Quelques point de la théorie des Ensembles*). The formal study of Lindelöf spaces was begun in 1921 by Kuratowski and Sierpinski (*La Théoréme de Boral-Lebesgue dans la Théorie des Ensembles Abstraits*). Lindelöf spaces were called finally compact spaces by authors in Russia, see for example, Alexandroff (*Some Results in the Theory of Topological Spaces*) (see [85]).

The idea of Lindelöf property came from studying compactness property. Since compactness is a very important property in topology and analysis, mathematicians have studied it widely. In compact spaces we deal with open covers to admit finite subcovers. After that compactness was generalized to countable compactness and sequentially compactness. The idea of Lindelöf property came later by dealing with open covers to admit countable subcovers.

Since the relationship between compactness and Lindelöfness are very strong, where every compact space is Lindelöf but not the converse, many properties of compact spaces were generalized onto Lindelöf spaces. Thus mathematicians first studied some generalizations of compact spaces such as: paracompactness, metacompactness, real compactness, nearly compactness, weakly compactness etc. Mathematicians called all these concepts covering properties. They generalized these concepts again to new covering properties such as nearly Lindelöf, almost Lindelöf, weakly Lindelöf, para-Lindelöf and other generalizations.

Many generalizations of Lindelöf spaces have been introduced by several authors for different reasons and purposes. Some of these generalizations depend on their definitions on open covers as in Lindelöf spaces. Others depend on regular open covers, regular closed covers, and regular covers which are introduced by Cammaroto and Lo Faro [6]. About generalizations of Lindelöf spaces that depend on open covers and regular open covers, Frolik [25] introduced the notion of weakly Lindelöf spaces, which were then studied by many authors; see for examples [9], [26], [84], [88] and [3]. After that, Balasubramaniam [2] introduced and studied nearly Lindelöf spaces that are between Lindelöf spaces and weakly Lindelöf spaces. Willard and Dissanayake [86] gave the notion of almost κ-Lindelöf spaces, that for $\kappa = \aleph_0$ are called almost Lindelöf spaces, and that are between nearly Lindelöf spaces and weakly Lindelöf spaces.

Moreover, Cammaroto and Santoro [7] studied some characterizations and relations among nearly Lindelöf, almost Lindelöf and weakly Lindelöf spaces. It was also introduced the notions of almost regular-Lindelöf, nearly regular-Lindelöf and weakly regular-Lindelöf spaces as a new generalizations of Lindelöf spaces that depend on regular covers. Cammaroto and Santoro have studied some characterizations of almost regular-Lindelöf spaces and left the study of the other