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Differential Game with Many Pursuers when Controls are Subjected to Coordinate-wise Integral Constraints

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ABSTRACT

In this paper, we study a differential game of many pursuers and one evader in $\mathbb{R}^2$. The motions of all players are simple. An integral constraint is imposed on each coordinate of the control functions of players. We say that pursuit is completed if the state of a pursuer coincides with that of the evader at some time. The pursuers try to complete the pursuit, and the evader tries to avoid this. Sufficient conditions for completion of the differential game were obtained. The strategies of the pursuers are constructed based on the current values of control parameter of the evader. Also an illustrative example is provided.

Keywords: Differential game, pursuit, control, strategy, coordinate-wise integral constraint.

1. Literature review

A considerable amount of literature has been published on differential games when integral constraints are imposed on control functions (see, e.g., Azimov (1975), Chikrii and Belousov (2010), Ibragimov et al. (2011), Ibragimov and Satimov (2012), Krasovskii (1968), Kuchkarov et al. (2013), Samatov (2013), Satimov (2003), Satimov et al. (1984) and Satimov et al. (1983)). Integral constraints are constraints on resource, energy etc. Most of the published works are devoted to pursuit differential games with integral constraints.

In the paper of Nikolskii (1969), the first direct method of Pontryagin Pontryagin (1988) was developed to pursuit differential games with integral constraints.

In the work of Chikrii and Belousov (2010), linear pursuit game problems with integral constraints were studied. The proposed approach is based on the method of resolving functions. Sufficient conditions were obtained to terminate the game, which is analog of the Pontryagin condition.

The pursuit differential games involving several objects took the attention of many authors. For examples, in the works of Abramyantz and Maslov (2004), Bannikov (2009), Blagodatskikh and Petrov (2009), Grigorenko (1990), Ivanov (1978), Ivanov and Ledyaev (1981), Petrosjan (1966), Petrov (1997), Petrov, Pshenichii (1976), Sakharov (2010), Jin and Qu (2010), Stipanovic et al. (2009), Vagin and Petrov (2001), differential games with many pursuers were studied under geometric constraints, whereas in the works of Ibragimov (1998), Ibragimov (2004), Ibragimov and Satimov (2012), Kuchkarov et al. (2013), Samatov (2013), Satimov (2003), Satimov et al. (1984), Satimov et al. (1983), differential games of many pursuers and one evader were studied under integral constraints.

A new approach was proposed by Kuchkarov et al. (2013) for solution of the linear pursuit-evasion differential game of many pursuers and one evader when duration of the game is fixed. In this paper, estimates for the value of the game from below and above were established.

In the work of Ibragimov and Salleh (2012), pursuit differential game of a group of pursuers and one evader is described by linear equations where admissible controls of players belong to the space $L_p(0,T)$. Resolving function method is used to construct strategies of the pursuers. Authors obtained sufficient conditions for solvability of pursuit problem.

The works of Satimov et al. (1984) and Satimov et al. (1983) are devoted to multi pursuer linear differential and discrete games. In particular, many pursuit differential games under integral constraints were studied in these papers. The control parameters of the pursuers u_i and that of the evader v are subjected to the following integral constraints

$$\int_0^\infty |u_i(s)|^2 ds \le \rho_i^2, \ i = 1, 2, \dots, m, \ \int_0^\infty |v(s)|^2 ds \le \sigma^2,$$

where ρ_i and σ are given positive numbers. The numbers ρ_i^2 and σ^2 are called control resources of the ith pursuer and the evader respectively. They obtained sufficient conditions in terms of the main matrix in the equation and control resources, for terminating the game. Note that one of the conditions is the inequality $\rho_1^2 + \rho_2^2 + \dots + \rho_m^2 > \sigma^2$.

A pursuit differential game of m pursuers and k evaders described by equations

$$\dot{x}_i = \varphi(t) u_i, \ x_i(0) = x_{i0}, \ i = 1, 2, \dots, m,$$
$$\dot{y}_j = \varphi(t) v_j, \ y_j(0) = y_{j0}, \ j = 1, 2, \dots, k,$$

$$\int_0^\infty |u_i(s)|^2 ds \le \rho_i^2, \ i = 1, 2, \dots, m, \ \int_0^\infty |v_j(s)|^2 ds \le \sigma_j^2, \ j = 1, 2, \dots, k.$$

was studied in Ibragimov and Satimov (2012), where $x_i, u_i, y_j, v_j \in \mathbb{R}^n$, u_i is the pursuers' control parameters, and v_j is the evaders' ones, and $\varphi(t)$ is assumed to be a scalar measurable function. Given a nonempty convex subset N of $\mathbb{R}^n$, $n \ge 2$, it is assumed that, in process of the game, all players move within the set N. Control functions of the players satisfy integral constraints. Under the condition

$$\rho_1^2 + \rho_2^2 + \dots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \dots + \sigma_k^2, \tag{1}$$

it was shown that pursuit can be terminated from any initial positions of the players in N. Note that the inequality (1) has the meaning that the total resource of the pursuers is greater than that of the evaders.

It should be noted that in almost all of the past research, the integral constraint on the control function of any player has the form

$$\int_0^\infty |u(s)|^2 ds \le \rho^2.$$

However, in Control Theory, in the case of geometric constraints, a special interest was shown in coordinate-wise constraints (see, for example, Lee and Markus (1967), Leitmann (1966), Ch.2). What is if coordinate-wise integral constraints are imposed on controls? This seems to be natural since each component of a control function may be separately subjected to integral constraint. In the paper of Ibragimov and Salleh (2012), in contrast to other works, a differential game was studied under coordinate-wise integral constraints of the form

$$\int_0^\infty u_1^2(s)ds \le \rho_1^2, \int_0^\infty u_2^2(s)ds \le \rho_2^2$$

where ρ_1, ρ_2 are given positive numbers. In this work, an evasion differential game of one evader from many pursuers was studied. Sufficient conditions of evasion were obtained and the strategy of the evader was constructed.

In the present paper, a differential game of several pursuers and one evader is studied in space $\mathbb{R}^2$. Similar to the work of Ibragimov and Salleh (2012), the control functions of the players are subjected to coordinate-wise integral constraints. We study, here, a pursuit game. We obtain a sufficient condition for solvability of pursuit problem in terms of control resources of players.

2. Statement of problem

We study a pursuit differential game of several pursuers and one evader which is described by the following differential equations:

$$\dot{x}_i = u_i, \; x_i(0) = x_{i0}, \tag{2}$$

$$\dot{y} = v, \; y(0) = y_0, \tag{3}$$

where $x_i, y, u_i, v \in \mathbb{R}^2$, $x_{i0} \neq y_0$, $u_i = (u_{i1}, u_{i2})$ is control parameter of the pursuer x_i, $i = 1, 2, \ldots, m$ and v is that of the evader y.

Definition 2.1. *Admissible control of the pursuer x_i is defined as the measurable function $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $t \ge 0$, subjected to coordinate-wise integral constraints*

$$\int_0^\infty u_{i1}^2(s)ds \le \rho_{i1}^2, \int_0^\infty u_{i2}^2(s)ds \le \rho_{i2}^2, \tag{4}$$

where ρ_{i1}, ρ_{i2}, $i = 1, 2, \ldots, m$, are given positive numbers.

Definition 2.2. *Admissible control of the evader y is defined as the measurable function $v(t) = (v_1(t), v_2(t))$, $t \ge 0$, subjected to constraints*

$$\int_0^\infty v_1^2(s)ds \le \sigma_1^2, \int_0^\infty v_2^2(s)ds \le \sigma_2^2, \tag{5}$$

where σ_1 and σ_2 are given positive numbers.

Definition 2.3. *A function*

$$U_i(t,y,v) = (U_{i1}(t,y,v), U_{i2}(t,y,v)),\ U_i : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2,$$

is referred to as the strategy of the pursuer x_i *if for an arbitrary admissible control of the evader* $v = v(t)$, $t \geq 0$, *the initial value problem*

$$\dot{x}_i = U_i(t,y,v),\ x_i(0) = x_{i0},$$
$$\dot{y} = v,\ y(0) = y(0),$$

has a unique solution $(x_i(\cdot), y(\cdot))$ *at* $v = v(t)$, $t \geq 0$, *and along this solution the following coordinate-wise integral constraints*

$$\int_0^\infty U_{i1}^2(t,y(t),v(t))dt \leq \rho_{i1}^2,\ \int_0^\infty U_{i2}^2(t,y(t),v(t))dt \leq \rho_{i2}^2$$

are satisfied.

By definition, pursuit is said to be completed in the game (2)-(5) from an initial position $\{y_0, x_{10}, \dots, x_{m0}\}$ if there exists a strategy of the pursuer such that for any admissible control of the evader, the equality $x_s(\tau) = y(\tau)$ occurs at some $\tau \geq 0$ and $s \in \{1, 2, \dots, m\}$.

We construct strategy for pursuers to complete the pursuit. Behavior of the evader is assumed to be any, that is, the evader applies any admissible control.

Problem. Find sufficient conditions on control resources of the players so that pursuit can be completed in the game (2)-(5).

3. Main result

In this section, we formulate and prove the main result of the paper. Denote

$$S_1 = \left\{ (\sigma_1^2, \sigma_2^2) | \sigma_2^2 < \sum_{i \in I(\sigma_1)} \rho_{i2}^2 \right\},\ S_2 = \left\{ (\sigma_1^2, \sigma_2^2) | \sigma_1^2 < \sum_{i \in I(\sigma_2)} \rho_{i1}^2 \right\},\ S = S_1 \cup S_2,$$

where

$$I(\sigma_1) = \{i | \sigma_1 < \rho_{i1}\},\ I(\sigma_2) = \{i | \sigma_2 < \rho_{i2}\}.$$

We'll prove the following statement.

Theorem. If $(\sigma_1^2, \sigma_2^2) \in S$, then pursuit can be completed in the game (2)-(5) from any initial position $\{y_0, x_{10}, \dots, x_{m0}\}$.

Proof. If $(\sigma_1^2, \sigma_2^2) \in S$, then either $(\sigma_1^2, \sigma_2^2) \in S_1$ or $(\sigma_1^2, \sigma_2^2) \in S_2$. To be specific consider the case $(\sigma_1^2, \sigma_2^2) \in S_1$. The same reasoning applies to the case $(\sigma_1^2, \sigma_2^2) \in S_2$. By definition of S_1, we have $\sigma_2^2 < \rho^2$, where $\rho = \left(\sum_{i \in I(\sigma_1)} \rho_{i2}^2\right)^{1/2}$. For simplicity we assume that $I(\sigma_1) = \{1, 2, \dots, l\}$.

Denote

$$\theta_i = \frac{\left(y_1^0 - x_{i1}^0\right)^2}{(\rho_{i1} - \sigma_1)^2}, \quad i = 1, 2, \dots, l.$$

Define the first coordinates of the pursuers' strategies as follows:

$$u_{i1}(t) = \begin{cases} \frac{1}{\theta_i}(y_1^0 - x_{i1}^0) + v_1(t), & 0 \le t \le \theta_i, \\ v_1(t), & t \ge \theta_i, \end{cases} \quad i = 1, 2, \dots, l. \tag{6}$$

It is not difficult to verify that

$$\int_0^\infty u_{i1}^2(t)dt \le \rho_{i1}^2.$$

Indeed, using the inequality $\int_0^\infty v_1^2(t)dt \le \sigma_1^2$ and definition of θ_i, yields

$$\begin{aligned} \int_0^\infty u_{i1}^2(t)dt &= \int_0^{\theta_i} u_{i1}^2(t)dt + \int_{\theta_i}^\infty u_{i1}^2(t)dt \\ &= \int_0^{\theta_i} \left| \frac{1}{\theta_i}(y_1^0 - x_{i1}^0) + v_1(t) \right|^2 dt + \int_{\theta_i}^\infty v_1^2(t)dt \\ &= \frac{1}{\theta_i}|y_1^0 - x_{i1}^0|^2 + \frac{2}{\theta_i}\int_0^{\theta_i} (y_1^0 - x_{i1}^0)v_1(t)dt + \int_0^\infty v_1^2(t)dt \\ &\le (\rho_{i1} - \sigma_1)^2 + \frac{2}{\theta_i}|y_1^0 - x_{i0}^0| \cdot \int_0^{\theta_i} |v_1(t)|dt + \sigma_1^2. \end{aligned} \tag{7}$$

Using the Cauchy-Schwartz inequality, we can estimate the integral on the right hand side of (7) as follows

$$\int_0^{\theta_i} |v_1(t)|dt \le \sqrt{\int_0^{\theta_i} 1^2 dt}\sqrt{\int_0^{\theta_i} v_1^2(t)dt} \le \sqrt{\theta_i}\sigma_1.$$

Then it follows from (7) and definition of θ_i that

$$\int_0^\infty u_{i1}^2(t)dt \le (\rho_{i1} - \sigma_1)^2 + \frac{2}{\theta_i}|y_1^0 - x_{i1}^0| \cdot \sqrt{\theta_i}\sigma_1 + \sigma_1^2 \le \rho_{i1}^2.$$

An easy computation shows that when the pursuer x_i applies (6) the equality $x_{i1}(t) = y_1(t)$ holds for all $t \geq \theta_i$ and $i = 1, 2, \ldots, l$.

Let
$$\tau_0 = \max_{i\in\{1,2,\ldots,l\}} \theta_i, \ \sigma_{2i} = \frac{\rho_{i2}}{\rho}\sigma_2, \ i = 1, 2, \ldots, l.$$

Clearly, $\sigma_{2i} < \rho_{i2}$ for all $i = 1, 2, \ldots, l$. Further, construct the second coordinates of the strategies of the pursuers. On the interval $[0, \tau_0]$, we set

$$u_{i2}(t) = 0, \ 0 \leq t \leq \tau_0, \ i = 1, 2, \ldots, l.$$

Let $\tau_1 = \tau_0 + |y_2(\tau_0) - x_{12}^0|^2/(\rho_{i2} - \sigma_{21})^2$. On the interval $[\tau_0, \infty]$, we define the second coordinate of the strategy of the first pursuer as follows

$$u_{12}(t) = \begin{cases} \dfrac{1}{\tau_1 - \tau_0}(y_2(\tau_0) - x_{12}^0) + v_2(t), & \tau_0 \leq t \leq \tau_1, \\ v_2(t), & t > \tau_1. \end{cases} \tag{8}$$

All other pursuers don't move on the time interval $[\tau_0, \tau_1]$, i.e.

$$u_{i2}(t) = 0, \ \tau_0 \leq t \leq \tau_1, \ i = 2, 3, \ldots l.$$

If
$$\int_{\tau_0}^{\tau_1} v_2^2(t)dt \leq \sigma_{21}^2 \tag{9}$$

then it is not difficult to show as above that

$$\int_{\tau_0}^{\tau_1} u_{12}^2(t)dt \leq \rho_{12}^2$$

and $x_{12}(\tau_1) = y_2(\tau_1)$, which implies the equality $x_1(\tau_1) = y(\tau_1)$ since $x_{i1}(t) = y_1(t)$ for all $t \geq \tau_0$, and $i = 1, 2, \ldots, l$.

Thus if the inequality (9) holds, then the strategy (8) of the pursuer x_1 guarantees completion of pursuit at time τ_1.

If the inequality (9) fails to hold, i.e.

$$\int_{\tau_0}^{\tau_1} v_2^2(t)dt > \sigma_{21}^2,$$

then we construct the strategies of the pursuers $x_2, \ldots, x_l$ inductively. Assume that
$$\int_{\tau_{j-1}}^{\tau_j} v_2^2(t)dt > \sigma_{2j}^2 \text{ for all } j = 1, 2, \ldots, k, \tag{10}$$

where k, $1 \leq k < 1$, is an integer. We construct the second coordinate of the strategy of the pursuer x_{k+1} as follows. Denote

$$\tau_{k+1} = \tau_k + \frac{|y_2(\tau_k) - x^0_{k+1,2}|^2}{(\rho_{k+1,2} - \sigma_{2,k+1})^2}.$$

Define

$$u_{k+1,2}(t) = \begin{cases} \dfrac{1}{\tau_{k+1} - \tau_k}(y_2(\tau_k) - x^0_{k+1,2}) + v_2(t), & \tau_k \leq t \leq \tau_{k+1}, \\ v_2(t), & t > \tau_{k+1}, \end{cases}$$

$$u_{i2}(t) = 0,\ \tau_k \leq t \leq \tau_{k+1},\ i \in \{1, 2, \ldots, l\} \backslash \{k+1\}.$$

Clearly, if

$$\int_{\tau_k}^{\tau_{k+1}} v_2^2(t)dt \leq \sigma^2_{2,k+1},$$

then pursuit is completed by the pursuer x_{k+1} at time τ_{k+1}. Therefore, we only need to show that the inequality

$$\int_{\tau_{s-1}}^{\tau_s} v_2^2(t)dt \leq \sigma^2_{2s}$$

holds at some integer s, $1 \leq s \leq l$. Indeed, if this is not so, we obtain from the inequalities (10) that

$$\int_0^\infty v_2^2(t)dt \geq \sum_{s=1}^{l} \int_{\tau_{s-1}}^{\tau_s} v_2^2(t)dt > \sum_{s=1}^{l} \sigma_2^2 = \sum_{s=1}^{l} \frac{\rho_{i2}^2}{\rho^2} \cdot \sigma_2^2 = \sigma_2^2,$$

which contradicts the second inequality in (5). Therefore, (10) fails to hold at some $j = s \in \{1, 2, \ldots, l\}$. Then pursuer x_s completes the pursuit. The proof of the theorem is complete.

The following example illustrates the theorem.

Example. Let $m = 5$ and $(\rho^2_{11}, \rho^2_{12}) = (1, 2)$, $(\rho^2_{21}, \rho^2_{22}) = (2, 3)$, $(\rho^2_{31}, \rho^2_{32}) = (2, 4)$, $(\rho^2_{41}, \rho^2_{42}) = (3, 6)$, $(\rho^2_{51}, \rho^2_{52}) = (5, 1)$. We construct the set $S = S_1 \cup S_2$ in the plane of resources (σ_1^2, σ_2^2). If the resource vector (σ_1^2, σ_2^2) of the evader belongs to S, then pursuit can be completed. First, we construct the set $S_1 = \left\{(\sigma_1^2, \sigma_2^2) | \sigma_2^2 < \sum_{i \in I(\sigma_1)} \rho^2_{i2}\right\}$, $I(\sigma_1) = \{i | \sigma_1 \leq \rho_{i1}\}$. Since $\rho^2_{11} = 1$, $\rho^2_{21} = 2$, $\rho^2_{31} = 2$, $\rho^2_{41} = 3$, $\rho^2_{51} = 5$, the set S_1 is constructed as follows.

If $0 < \sigma_1^2 \leq \rho_{11}^2 = 1$, then $I(\sigma_1) = \{1, 2, 3, 4, 5\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|0 \leq \sigma_1^2 \leq 1,\ \sigma_2^2 < \rho_{12}^2+\rho_{22}^2+\rho_{32}^2+\rho_{42}^2+\rho_{52}^2\} = \{(\sigma_1^2, \sigma_2^2)|0 \leq \sigma_1^2 \leq 1,\ \sigma_2^2 < 16\}. \quad (11)$$

If $1 < \sigma_1^2 \leq \rho_{21}^2 = 2$, then $I(\sigma_1) = \{2, 3, 4, 5\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|1 \leq \sigma_1^2 \leq 2,\ \sigma_2^2 < \rho_{22}^2+\rho_{32}^2+\rho_{42}^2+\rho_{52}^2\} = \{(\sigma_1^2, \sigma_2^2)|1 \leq \sigma_1^2 \leq 2,\ \sigma_2^2 < 14\}. \quad (12)$$

If $2 < \sigma_1^2 \leq \rho_{41}^2 = 3$, then $I(\sigma_1) = \{4, 5\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|2 \leq \sigma_1^2 \leq 3,\ \sigma_2^2 < \rho_{42}^2 + \rho_{52}^2\} = \{(\sigma_1^2, \sigma_2^2)|2 \leq \sigma_1^2 \leq 3,\ \sigma_2^2 < 7\}. \quad (13)$$

If $3 < \sigma_1^2 \leq \rho_{51}^2 = 5$, then $I(\sigma_1) = \{5\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|3 \leq \sigma_1^2 \leq 5,\ \sigma_2^2 < \rho_{52}^2\} = \{(\sigma_1^2, \sigma_2^2)|3 \leq \sigma_1^2 \leq 5,\ \sigma_2^2 < 1\}. \quad (14)$$

Note that $I(\sigma_1) = \varnothing$ if $\sigma_1^2 > \rho_{51}^2 = 5$. The set S_1 is now union of the sets (12)-(14).

Similarly, we can construct the set $S_2 = \left\{(\sigma_1^2, \sigma_2^2)|\sigma_1^2 < \sum_{i \in I(\sigma_2)} \rho_{i1}^2\right\}$, where $I(\sigma_2) = \{i|\sigma_2 \leq \rho_{i2}\}$.

If $0 < \sigma_2^2 \leq \rho_{52}^2 = 1$, then $I(\sigma_2) = \{1, 2, 3, 4, 5\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|\sigma_1^2 < \rho_{11}^2+\rho_{21}^2+\rho_{31}^2+\rho_{41}^2+\rho_{51}^2,\ 0 < \sigma_2^2 \leq 1\} = \{(\sigma_1^2, \sigma_2^2)|\ \sigma_1^2 < 13,\ 0 < \sigma_2^2 \leq 1\}. \quad (15)$$

If $1 < \sigma_2^2 \leq \rho_{12}^2 = 2$, then $I(\sigma_2) = \{1, 2, 3, 4\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|\sigma_1^2 < \rho_{11}^2+\rho_{21}^2+\rho_{31}^2+\rho_{41}^2,\ 1 < \sigma_2^2 \leq 2\} = \{(\sigma_1^2, \sigma_2^2)|\ \sigma_1^2 < 8,\ 1 < \sigma_2^2 \leq 2\}. \quad (16)$$

If $2 < \sigma_2^2 \leq \rho_{22}^2 = 3$, then $I(\sigma_2) = \{2, 3, 4\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|\sigma_1^2 < \rho_{21}^2 + \rho_{31}^2 + \rho_{41}^2,\ 2 < \sigma_2^2 \leq 3\} = \{(\sigma_1^2, \sigma_2^2)|\ \sigma_1^2 < 7,\ 2 < \sigma_2^2 \leq 3\}. \quad (17)$$

If $3 < \sigma_2^2 \leq \rho_{32}^2 = 4$, then $I(\sigma_2) = \{3, 4\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2)|\sigma_1^2 < \rho_{31}^2 + \rho_{41}^2,\ 3 < \sigma_2^2 \leq 4\} = \{(\sigma_1^2, \sigma_2^2)|\ \sigma_1^2 < 5,\ 3 < \sigma_2^2 \leq 4\}. \quad (18)$$

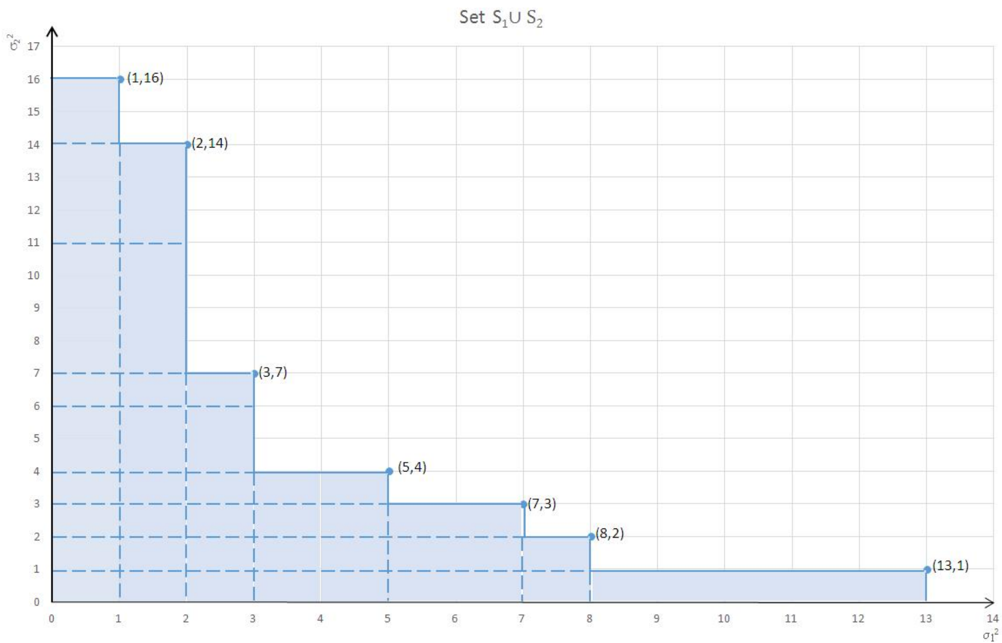

Figure 1: The set S corresponding to the pursuers' resources $(\rho_{11}^2, \rho_{12}^2) = (1, 2)$, $(\rho_{21}^2, \rho_{22}^2) = (2, 3)$, $(\rho_{31}^2, \rho_{32}^2) = (2, 4)$, $(\rho_{41}^2, \rho_{42}^2) = (3, 6)$, $(\rho_{51}^2, \rho_{52}^2) = (5, 1)$.

If $4 < \sigma_2^2 \le \rho_{42}^2 = 6$, then $I(\sigma_2) = \{4\}$ and so,

$$\{(\sigma_1^2, \sigma_2^2) | \sigma_1^2 < \rho_{41}^2,\ 4 < \sigma_2^2 \le 6\} = \{(\sigma_1^2, \sigma_2^2) |\ \sigma_1^2 < 3,\ 4 < \sigma_2^2 \le 6\}. \qquad (19)$$

Note that $I(\sigma_2) = \varnothing$ if $\sigma_2^2 > 6$. The set S_2 is now union of the sets (15)-(19).

We can now see that the set $S = S_1 \cup S_2$ is a polygon with the successive vertices at the points

$$(0,0), (13,0), (13,1), (8,1), (8,2), (7,2), (7,3), (5,3), (5,4),$$

$$(3,4), (3,7), (2,7), (2,14), (1,14), (1,16), (0,16)$$

(see Fig. 1).

4. Conclusion

We have studied a pursuit differential game of many pursuers. The main difference of this game from those studied by other researchers is that control functions of all players are subjected to coordinate-wise integral constraints. It was shown earlier (see Example 1, Ibragimov and Salleh (2012)) that inequalities $\rho_{1i}^2 + \rho_{2i}^2 + \ldots + \rho_{mi}^2 > \sigma_i^2$, $i = 1, 2$, are not sufficient to complete the pursuit in the game. In the present paper, a sufficient condition of completion of pursuit has been obtained.

The problem is open for further investigation if the hypothesis of the theorem is not satisfied. Further studies can be done to obtain complete solution for the pursuit game problem under coordinate-wise integral constraints.

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