UNIVERSITI PUTRA MALAYSIA

CLASSIFICATION OF A SUBCLASS OF FILIFORM LEIBNIZ ALGEBRAS

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CLASSIFICATION OF A SUBCLASS OF FILIFORM LEIBNIZ ALGEBRAS

By

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Thesis Submitted to the School of Graduate Studies,
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DEDICATIONS

My incomparable parents
and invaluable teachers in all realms of my studies.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science.

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ABDULKAREEM ABDULAFEEZ OLALEKAN

May 2014

Chair: Prof. Isamiddin S. Rakhimov, Ph.D.

Faculty: Institute for Mathematical Research

This study centers on isomorphism classes and invariants of a subclass of filiform Leibniz algebras over complex field. The subclass of filiform Leibniz algebras considered arises from naturally graded filiform Lie algebras. It has been denoted by TLb_n in a fixed dimension n. It is noted that n-dimensional filiform Lie algebras are in TLb_n. The intent with this study is to find the classification of third class of filiform Leibniz algebras in dimensions 7 and 8. The classification is carried out by first choosing adapted basis, then construct appropriate multiplication table of the said basis. From the multiplication table, isomorphism criterion is set up using adapted linear transformation and elementary base change. With respect to the condition on the structure constants in adapted basis, different disjoint subsets of algebras in TLb_7 and TLb_8 are obtained. Some of these subsets are single orbits while others are represented as a union of parametric family of orbits. In parametric families of case, the invariants that characterize the parameter in the orbits are given. The filiform Lie algebras in each dimension are specified.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah master sains

KLASIFIKASI SATU SUBKELAS FILIFORM ALJABAR LEIBNIZ

Oleh

ABDULKAREEM ABDULAFEEZ OLALEKAN

Mei 2014

Pengerusi: Prof. Isamiddin S. Rakhimov, Ph.D.

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Kajian ini tertumpu kepada kelas isomorfisma dan invarian daripada subkelas filiform aljabar Leibniz dalam medan kompleks. Subkelas filiform aljabar Leibniz dianggap terbit secara semulajadi daripada gred filiform aljabar Lie. Ia telah ditandakan dengan simbol TLb_n dalam dimensi n tetap. Ia menyatakan bahawa n dimensi filiform aljabar Lie adalah dalam TLb_n. Tujuan kajian ini adalah untuk mencari pengelesan kelas ketiga aljabar Leibniz filiform dalam dimensi tujuh dan lapan. Klasifikasi yang pertama adalah memilih basis yangdisesuaikan , kemudian dibina satu jadual pendaraban sesuai dengan asas tersebut. Daripada jadual pendaraban, kriteria isomorfisma dibina menggunakan transformasi linear disesuaikan dan perubahan asas. Berkenaan dengan keadaan di pemalar struktur asas disesuaikan, subset yang berbeza tidak sama pada aljabar dalam TLb_7 dan TLb_8 yang diperolehi. Sebahagian daripada subset adalah orbit tunggal manakala yang lain adalah diwakili oleh satu kesatuan keluarga parametrik daripada orbit. Dalam kes keluarga parametrik ini, invarian-invarian yang mencirikan parameter dalam orbit-orbit telah diberikan. Filiform aljabar Lie adalah dalam setiap dimensi yang dinyatakan.

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I certify that a Thesis Examination Committee has met on 9 May 2014 to conduct the final examination of Abdulkareem Abdulafeez Olalekan on his thesis entitled "Classification of a Subclass of Filiform Leibniz Algebras" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science.

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LIST OF ABBREVIATIONS

Lb_n	The class of n-dimensional Filiform Leibniz Algebra
FLb_n	First class of n-dimensional First Class Filiform Leibniz Algebra
SLb_n	Second class of n-dimensional Second Class Filiform Leibniz Algebra
TLb_n	Third class of n-dimensional Third Class Filiform Leibniz Algebra

CHAPTER 1

INTRODUCTION

This chapter is meant to give a bird's eye view of the contents of this thesis. The chapter includes; the introductory section, research questions, and research objectives of this study.

1.1 Introduction

An algebra over a field $\mathbb{K}$ is a vector space over $\mathbb{K}$ equipped with a bilinear binary operation. The problem of describing finite dimensional algebras up to isomorphism is a fundamental problem in structure theory of algebras. This research is focused on a part of structure theory of algebra usually referred to as "classification problem". Classification problem is solved for finite dimensional algebras by setting up a list of subsets which represent each isomorphism class uniquely. The list could be interpreted as a parametrization of the orbit space associated with the canonical action of the linear group, $GL(V)$, on the space $Hom(V \otimes V, V)$, where V is an n-dimensional vector space. In this perspective, the classification problem for finite dimensional algebras relates to questions in invariant theory.

Leibniz algebras were introduced as a non-antisymmetric analogue of Lie algebras by J. L. Loday in 1993. Like Malcev algebras, pre-Lie algebras, Leibniz algebras is another form of generalization of Lie algebras. Since its introduction, Leibniz algebras applications have been found in various fields of Mathematics like Differential geometry, Homology theory, where its introduction was initiated, Quantum Physics etc.

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of an arbitrary Leibniz algebra, the table of multiplication is defined by the products of basis elements. The products $[e_i, e_j] = \sum\limits_{k=0}^{n} \gamma_{ij}^k e_k$ completely determine the products of arbitrary elements of the algebra. The coefficients, γ_{ij}^k, are called structure constants of the algebra relative to the basis $\{e_1, e_2, \cdots, e_n\}$. Thus, classification problem can be reduced to the description of the structure constants up to a non-degenerate basis transformation. Since such description is generally difficult to handle, different methods of handling classification problem have been proposed. The present study employed one of these methods to classify a subclass of filiform Leibniz algebras.

In this thesis, we are concerned with isomorphism classes and invariants of a subclass of filiform Leibniz algebras over complex field. The subclass of filiform Leibniz algebras considered arises from naturally graded filiform Lie algebras. In a fixed dimension n, it has been denoted by TLb_n. It is noted that n-dimensional filiform Lie algebras are in TLb_n. Rakhimov and Hassan (2011b) studied isomorphism classes and invariants of this subclass in dimensions 5 and 6. This study extends the result of Rakhimov and Hassan (2011b) to dimensions 7 and 8 with slight modification to various tools used therein. What is done when isomorphism classes are given is nothing but classification. The classification is carried out by first choosing an adapted basis such that the table multiplication of an algebra has the most convenient form. Using adapted linear transformation and elementary base change, the study of all transformation of the adapted basis is reduced to simple ones. Thereafter, the relation between structure constants in initial and transformed bases is established and this relation is referred to as isomorphism criterion. With respect to this criterion, we present the list of pairwise non-isomorphic algebras in TLb_7 and TLb_8. Some of the representatives of these algebras are single orbits while others are represented as a union of parametric family of orbits. In parametric family of orbits case, the invariants that characterize the parameter in the orbits are given. The representatives of filiform Lie algebras in each dimension are specified.

In what follows, we give a brief outline of the thesis. The first chapter gives the introduction, research questions, aims and objectives of the research. In Chapter two, literature review of this study is given. Chapter three provides the basic constructions needed in this study as well as the methodology employed that enables us to answer all our research questions. Chapters four and five contain the main results of the study. The last chapter gives conclusions and recommendations for further research.

1.2 Research Questions

1. Construct the table of multiplication for TLb_7 and TLb_8?
2. What condition(s) must be satisfied for any two algebras in TLb_7 and TLb_8 to be isomorphic?
3. How many isomorphism classes are there in the list of representatives of algebras in TLb_7 and TLb_8?
4. What condition(s) must be satisfied for any two algebras in the list of isomorphism classes in TLb_7 and TLb_8 to be isomorphic?
5. Which of the isomophism classes are represented by filiform Lie algebras in each dimension considered?

1.3 Research Objectives

The aims and objectives of this study are:

1. To give the multiplication table of algebra in TLb_7 and TLb_8,
2. To give the isomorphism criterion for algebras in TLb_7 and TLb_8,
3. To give the list of isomorphism classes in TLb_7 and TLb_8
4. To give the set of invariant functions to distinguish the orbits in TLb_7 and TLb_8,
5. To give the list of representatives of filiform Lie algebras in each dimension and compared with the corresponding result for filiform Lie algebras obtained by Ancochéa-Bermudez (1988) and Gómez and Khakimdjanov (1998).

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