UNIVERSITI PUTRA MALAYSIA

NUMERICAL EVALUATION OF CAUCHY TYPE SINGULAR INTEGRALS USING MODIFICATION OF DISCRETE VORTEX METHOD

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FS 2007 30

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By

MOHAMMAD ABDULKAWI MAHIUB

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirement for the Degree of Master of Science

March 2007

DEDICATION

This thesis is dedicated to all my family members especially my father Abdulkawi Mahiub

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of requirement for the degree of Master of Science

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March 2007

Chairman: Nik Mohd Asri Nik Long, PhD

Faculty : Science

In this thesis, characteristic singular integral equations of Cauchy type

$$\int_L \frac{\varphi(t)}{x-t}\,dt = f(x), \quad x \in L, \tag{1}$$

where L is open or closed contour, are examined.

The analytical solutions for equation (1) are described. Some examples of solution for certain functions $f(x)$ are given.

A quadrature formula for evaluation of Cauchy type singular integral (SI) of the form

$$\int_{-1}^{1} \frac{\varphi(t)}{x-t}\,dt, \quad -1 < x < 1, \tag{2}$$

is constructed with equal partitions of the interval $[-1,1]$ using modification discrete vortex method (MMDV), where the singular point x is considered in the middle of one of the intervals $[t_j, t_{j+1}]$, $j=1,\ldots, n$.

It is known that the bounded solution of equation (1) when $L=[-1,1]$ is

$$\varphi(x) = \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}\,(t-x)}\,dt, \quad -1 < x < 1. \tag{3}$$

A quadrature formula is constructed to approximate the SI in (3) using MMDV and linear spline interpolation functions, where the singular point x is assumed to be at any point in the one of the intervals $[t_j, t_{j+1}]$, $j=1,\ldots, n$.

The estimation of errors of constructed quadrature formula are obtained in the classes of functions $C^1[-1,1]$ and $H^\alpha(A,[-1,1])$ for SI (2) and $H^\alpha(A,[-1,1])$ for (3). For SI (2), the rate of convergence is improved in the class $C^1[-1,1]$, whereas in the class $H^\alpha(A,[-1,1])$, the rate of convergence of quadrature formula is the same of that of discrete vortex method (MDV).

FORTRAN code is developed to obtain numerical results and they are presented and compared with MDV for different functions $f(t)$. Numerical experiments assert the theoretical results.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENYELESAIAN BERANGKA KAMIRAN SINGULAR JENIS CAUCHY MENGGUNAKAN PENGUBAHSUAIAN KAEDAH DISKRIT VORTEKS

Oleh

MOHAMMAD ABDULKAWI MAHIUB

March 2007

Pengerusi: Nik Mohd Asri Nik Long, PhD

Fakulti : Sains

Tesis ini mengkaji persamaan kamiran singular cirian jenis Cauchy iaitu

$$\int_L \frac{\varphi(t)}{x-t}\,dt = f\left(x\right), \quad x \in L, \tag{1}$$

dengan L adalah kontur terbuka dan tertutup.

Penyelesaian analisis untuk persamaan (1) dihuraikan. Beberapa contoh penyelesaian ditunjukkan untuk fungsi $f(x)$ tertentu diberikan.

Formula kuadratur untuk menilai kamiran singular (SI) jenis Cauchy berbentuk

$$\int_{-1}^{1} \frac{\varphi(t)}{x-t}\,dt, \quad -1 < x < 1, \tag{2}$$

dibina dengan partisi sama bagi selang $[-1,1]$ menggunakan kaedah pengubahsuaian diskrit vorteks (MMDV), dengan titik singular x dipertimbangkan berada di tengah adalah satu selang $[t_j, t_{j+1}]$, $j=1,\ldots, n$.

Diketahui bahawa penyelesaian terbatas bagi persamaan (1) dengan $L=[-1,1]$ ialah

$$\varphi(x) = \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}\,(t-x)} dt, \quad -1 < x < 1. \tag{3}$$

Formula kuadratur dibina untuk menganggarkan SI (3) menggunakan MMDV dan fungsi interpolasi splin linear, dengan titik singular x diandaikan berada pada sebarang titik dalam adalah satu selang $[t_j, t_{j+1}]$, $j=1,\ldots,n$.

Penganggaran ralat bagi formula kuadratur yang terbina diperolehi dalam kelas fungsi $C^1[-1,1]$ dan $H^{\alpha}(A,[-1,1])$ untuk SI (2) dan $H^{\alpha}(A,[-1,1])$ untuk (3). Untuk SI (2), kadar penumpuan dipertingkatkan dalam kelas $C^1[-1,1]$, manakala untuk kelas $H^{\alpha}(A,[-1,1])$, kadar penumpuan adalah sama dengan Kaedah Diskrit Vorteks (MDV).

Kod FORTRAN dibangunkan untuk mendapatkan keputusan berangka dan ianya dipersembahkan dan dibandingkan dengan MDV untuk fungsi $f(t)$ yang berbeza. Eksperimen berangka mengukuhkan keputusan yang diperolehi secara teori.

ACKNOWLEDGEMENTS

Firstly, praise to Allah (S. W. T) for given me enough courage, strength and patience to complete my study.

I would like to express my deepest gratitude to my supervisory committee, Dr. Zainnidin K. Eshkovatov and Dr. Nik Mohd Asri Nik Long for the guidance, patience, invaluable advice, motivation, encouragement and precious support during the period of my study.

I would also like to thank all members of department of Mathematics, faculty of Science, University Putra Malaysia, especially, Head of Department Associate professor Dr. Mohd Rizam Abu Bakar, Professor Dr. Mohammed Suleiman, Associate Professor Dr. Bekbaev Ural Djumaevich, Associate professor Dr. Adem Kilicman, and Associate Professor Dr. Mat Rofa Ismail for their help.

I would also like to thank all members of Aden Community college, Aden Yemen, especially dean of the college and all members of department of Mathematics for their support and motivation.

My thanks also go to all my friends and colleagues in Yemen and Malaysia for their friendship and help.

Special thanks are for all my family for their patience and support through the period of my study.

I certify that an Examination Committee has met on 28 March 2007 to conduct the final examination of Mohammad Abdulkawi Mahiub Abdalmoghny Alshamery on his Master of Science thesis entitled “Numerical Evaluation of Cauchy Type Singular Integrals Using Modification of Discrete Vortex Method” in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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This thesis submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee are as follows:

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

MOHAMMAD ABDULKAWI MAHIUB

Date: 2 MAY 2007

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LIST OF ABBREVIATIONS

SIE	: Singular integral equation
SI	: Singular integral
MDV	: Discrete vortex method
MMDV	: Modification of Discrete Vortex Method
EXACT	: Exact solution
MDV (4.46)	: Approximate solution obtained by (4.46).
MDV(4.47)	: Approximate solution obtained by (4.47).
MMDV(4.20)	: Approximate solution obtained by (4.20).
MMDV(4.30)	: Approximate solution obtained by (4.30).
ERR.MDV(4.46)	: Errors of MDV (4.46).
ERR.MDV(4.47)	: Errors of MDV (4.47).
ERR.MMDV(4.20)	: Errors of quadrature formula (4.20).
ERR.MMDV(4.30)	: Errors of quadrature formula (4.30).

CHAPTER I

INTRODUCTION

Historical introduction

Integral equation containing integrals, in the sense of the Cauchy principle value, with integrands having a singularity in the domain of integration is called singular integral equations. Singular integral equation was introduced in the first decade of the 20th century in connection with two quite different problems:

1) Hilbert encountered singular integral equations in a certain boundary problem of the theory of analytic functions.

2) Poincare encountered singular integral equations in the general theory of tides.

The theory of singular integral equations was systematically developed in the third and fourth decade of 20th century due to French mathematician, Giraud and Soviet mathematicians, Muskhelishvili, Gakhov and Vekua.

The investigations of Muskhelishvili and Vekua, based on the properties of Cauchy type integral and Riemann-Hilbert's problems, brought a number of interesting and important results in the theory of analytic function, and partial differential equations, as well as in problem of aerodynamics and the theory of elasticity. These investigations are presented in an excellent monograph of Muskhelishvili (1953).

Definition 1.1: L is said to be a smooth open-ended contour, if L can be defined by the parametric relations:

$$x = x(s), \qquad y = y(s), \qquad s_a \le s \le s_b \tag{1.1}$$

where s_a and s_b are finite constant; $x(s)$, $y(s)$ are continuously differentiable functions on $[s_a, s_b]$; the derivatives $x'(s)$, $y'(s)$ cannot be both equal to zero at the same point; and assume that different values of the parameter s correspond to different points of the curve L. The relation $t(s) = x(s) + iy(s)$ for the point s of the curve L establishes a one-to-one correspondence between $t_0 \in L$ and $s \in [s_a, s_b]$, and we have $t'(s) = x'(s) + iy'(s)$.

Definition 1.2: L is said to be a smooth closed contour, if L is a smooth contour such that

$$x(s_a) = x(s_b), \qquad y(s_a) = y(s_b), \tag{1.2}$$

$$x'(s_a + 0) = x'(s_b - 0), \qquad y'(s_a + 0) = y'(s_b - 0). \tag{1.3}$$

Definition 1.3: A curve is said to be piecewise smooth (Figure 1.1) if it consists of finitely many smooth open–ended-curves having no points in common except, end-points. Such a curve is said to have only angular nodes if the angle between any two curves entering each node is different from zero, i.e., the node cannot be a cuspidal point (Lifanov et al., 2004).

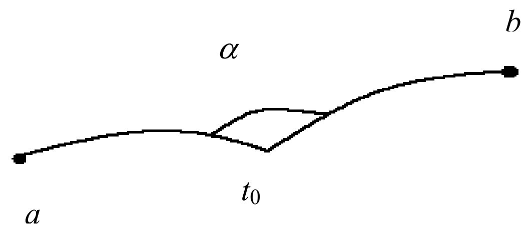

Figure 1.1: Piecewise smooth curve

Definition 1.4: A function $\varphi(t)$ defined on a set D is said to satisfy the Hölder condition with exponent α, if for any $t_1, t_2 \in D$, the inequality

$$|\varphi(t_2) - \varphi(t_1)| \le A\, |t_2 - t_1|^{\alpha} \tag{1.4}$$

holds with constant $A \geq 0$ and $0 < \alpha \leq 1$. These constants are respectively called the coefficient and the exponent in the Hölder condition. We simply say that the function $\varphi(t)$ satisfies the H-condition or belongs to the class H on the set D. Such a function φ is also said to be Hölder continuous (Kanwal, 1997).

We write $\varphi(t) \in H(\alpha)$ or $\varphi(t) \in H^{\alpha}(A, D)$.

Note that the inclusion $\varphi(t) \in H^{\alpha}(A, D)$ implies that $|\varphi(t)| \in H^{\alpha}(A, D)$.

Definition 1.5: A function $\varphi(t_1, \ldots, t_n)$ defined for $(t_1, \ldots, t_n) \in D$ is said to be of class $H(\alpha_1, \ldots, \alpha_n)$ on the set D, if for any points $(t''_1, \ldots, t''_n)$, $(t'_1, \ldots, t'_n) \in D$, the inequality

$$|\varphi(t''_1, \ldots, t''_n) - \varphi(t'_1, \ldots, t'_n)| \le A_1\, |t''_1 - t'_1|^{\alpha_1} + \ldots + A_n\, |t''_n - t'_n|^{\alpha_n}, \tag{1.5}$$

holds with constants $A_j \geq 0$, $0 < \alpha \leq 1$, $j = 1, 2, \ldots, n$.

Definition 1.6: A function $\varphi(t)$ belongs to the class H^* on a piecewise smooth contour L if

$$\varphi(t) = \frac{\varphi^*(t)}{P_L^V(t)}, \qquad P_L^V(t) = \prod_{k=1}^{p} |t - c_k|^{V_k} \tag{1.6}$$

where $\varphi^*(t) \in H_0$ on L, i.e., it belongs to the class H on every smooth piece of the contour L; $0 \leq \nu_k < 1$; and c_k, $k = 1, 2, \ldots, P$, are nodes of the contour L.

Cauchy type singular integral and Cauchy principal value

Definition 1.7: Let t_0 be a point on contour L outside its nodes. Consider a circle with center t_0 and small radius ε >0 that intersects L at two points t_1 and t_2 (Figure1.2). Denote by λ the arc $\overset{\frown}{t_1 t_2} \subset L$. If the integral

$$\int_{L/\ell} \frac{\varphi(t)}{t-t_0} dt, \qquad (1.7)$$

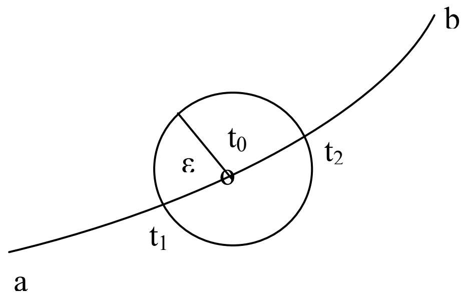

Figure 1.2: Cauchy principal value of the singular integral

has a finite limit $\Phi(t_0)$ as $\varepsilon \to 0$, this limit is called the Cauchy principal value of the singular integral,

$$\Phi(t_0) = \lim_{\varepsilon \to 0} \int_{L\setminus\ell} \frac{\varphi(t)}{t-t_0} dt, \qquad (1.8)$$

and it is denoted by (Kanwal, 1997)

$$\overset{*}{\int_L} \frac{\varphi(t)}{t-t_0} dt. \qquad (1.9)$$

Consider the special case of a singular integral on the segment L=[a, b] of the real axis OX. Then, formula (1.8) reads

$$\Phi(t_0)=\lim_{\varepsilon\to 0}\int_{L\setminus\ell}\frac{\varphi(t)}{t-t_0}dt=\lim_{\varepsilon\to 0}\left[\int_a^{t_0-\varepsilon}\frac{\varphi(t)}{t-t_0}dt+\int_{t_0+\varepsilon}^{b}\frac{\varphi(t)}{t-t_0}dt\right]$$

$$=\int_a^{*b}\frac{\varphi(t)}{t-t_0}dt,\quad a<t_0<b \tag{1.10}$$

The general formula for changing the order of two Cauchy principal integrals can be written as

$$\int_L^{*}\frac{d\tau}{\tau-t}\int_L^{*}\frac{K(\tau,\tau_1)}{\tau_1-\tau}d\tau_1=\int_L^{*}d\tau_1\int_L^{*}\frac{K(\tau,\tau_1)}{(\tau-t)(\tau_1-\tau)}d\tau-\pi^2K(t,t). \tag{1.11}$$

This is called Poincare-Bertrand formula (Andrei and Alexander, 1998).

We commence the investigation of the problem of existence of the singular integral with the simplest case

$$\int_L^{*}\frac{dt}{t-t_0}. \tag{1.12}$$

Denote by λ the part of the contour L cut out by the circle with center at t_0 whose radius ε>0 and take the integral over the remaining arc (Figure 1.3).

Then

$$\lim_{\varepsilon\to 0}\int_{L/\ell}\frac{dt}{t-t_0}=\lim_{\varepsilon\to 0}\left(\ln(t-t_0)\big|_a^{t_1}+\ln(t-t_0)\big|_{t_2}^{b}\right)$$

$$=\ln\frac{b-t_0}{a-t_0}+\lim_{\varepsilon\to 0}\ln\frac{t_1-t_0}{t_2-t_0} \tag{1.13}$$

where a and b are the end points of the contour L (Zabreyko et al.,1975). Note that

$$\ln\frac{t_1-t_0}{t_2-t_0}=\ln\left|\frac{t_1-t_0}{t_2-t_0}\right|+i\left[\arg(t_1-t_0)-\arg(t_2-t_0)\right], \tag{1.14}$$

and

$$|t_1 - t_0| = |t_2 - t_0|. \tag{1.15}$$

Due to (1.14) and (1.15) yields

$$\lim_{\varepsilon \to 0} \ln \frac{t_1 - t_0}{t_2 - t_0} = i\,\pi \tag{1.16}$$

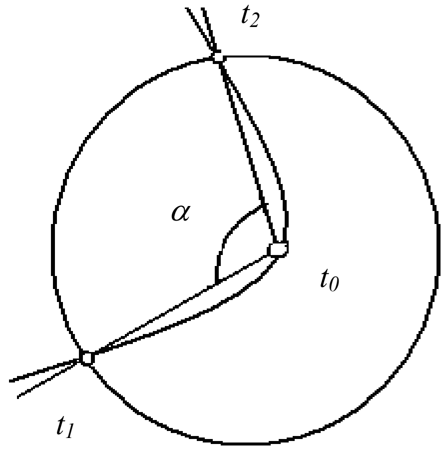

Figure 1.3: The angel between $t_1\, t_0$ and $t_2\, t_0$

and consequently

$$\int_L^* \frac{dt}{t - t_0} = \ln \frac{b - t_0}{a - t_0} + i\pi. \tag{1.17}$$

The later integral can also be represented in the form

$$\int_L^* \frac{dt}{t - t_0} = \ln \frac{b - t_0}{t_0 - a}. \tag{1.18}$$

If the contour L is closed, then

$$\int_L^* \frac{dt}{t - t_0} = i\,\pi. \tag{1.19}$$

Now consider the more general integral

$$\int_a^{*b} \frac{\varphi(t)}{t - t_0} dt, \tag{1.20}$$

where $\varphi(t)$ satisfies the Hölder condition and a and b are the end points of contour L.

In this case the integral (1.20) can easily be reduced to (1.12) by representing it in the form

$$\int_a^{*b} \frac{\varphi(t)}{t - t_0} dt = \int_a^{*b} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt + \varphi(t_0) \int_a^{*b} \frac{dt}{t - t_0}. \tag{1.21}$$

Substituting (1.18) into (1.21) yields

$$\int_a^{*b} \frac{\varphi(t)}{t - t_0} dt = \int_a^{*b} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt + \varphi(t_0) \ln \frac{b - t_0}{t_0 - a}. \tag{1.22}$$

It can be shown that the singular integral (1.20) exists if $\varphi(t)$ satisfies the Hölder condition (Davis and Rabinowitz, 1984).

Analytical Solution of the Cauchy type singular integral equation in a complex plane

In this section we attempt to solve the integral equation of the second kind

$$a\varphi(x) = f(x) - \frac{b}{\pi i} \int_L^{*} \frac{\varphi(t)}{t - x} dt \tag{1.23}$$

where a and b are constants, $\varphi(t)$ satisfy the Hölder condition, and L is a regular closed contour (Kanwal, 1997).

To find the solution for the closed contour, let us write (1.23) in the operator form

$$L\varphi_x = a\varphi(x) + \frac{b}{\pi i} \int_L^{*} \frac{\varphi(t)}{t - x} dt = f(x) \tag{1.24}$$

and define the adjoint operator

$$Mg_x = ag(x) - \frac{b}{\pi i}\int_L^* \frac{g(t)}{t-x}dt. \tag{1.25}$$

From (1.24) and (1.25) we obtain

$$ML\varphi = a\left(a\varphi(x) + \frac{b}{\pi i}\int_L^* \frac{\varphi(t)}{t-x}dt \right)$$

$$-\frac{b}{\pi i}\int_L^* \frac{dt}{t-x}\left(a\varphi(t) + \frac{b}{\pi i}\int_L^* \frac{\varphi(\tau)}{\tau-t}d\tau \right)$$

$$= af(x) - \frac{b}{\pi i}\int_L^* \frac{f(t)}{t-x}dt = Mf \tag{1.26}$$

Due to formula (1.11) and using (1.19) we obtain

$$\int_L^* \frac{dt}{t-x}\int_L^* \frac{\varphi(\tau)}{\tau-t}d\tau = \int_L^* d\tau \int_L^* \frac{\varphi(\tau)dt}{(t-x)(\tau-t)} - \pi^2\varphi(x)$$

$$= \int_L^* \frac{\varphi(\tau)d\tau}{\tau-x}\left(\int_L^* \frac{dt}{t-x} + \int_L^* \frac{dt}{\tau-t} \right) - \pi^2\varphi(x)$$

$$= -\pi^2\varphi(x). \tag{1.27}$$

Substituting (1.27) into (1.26) yields

$$a^2\varphi(x) - b^2\varphi(x) = af(x) - \frac{b}{\pi i}\int_L^* \frac{f(t)}{t-x}dt.$$

Thus the solution of equation (1.23) is

$$\varphi(x) = \frac{a}{a^2-b^2}f(x) - \frac{b}{(a^2-b^2)\pi i}\int_L^* \frac{f(t)}{t-x}dt, \tag{1.28}$$

where it is assumed that $a^2 - b^2 \neq 0$.

The solution of the Cauchy-type singular integral equation of the first kind

$$f(x) = \frac{1}{\pi i} \int_L^* \frac{\varphi(t)}{t-x} dt, \tag{1.29}$$

follows by setting a=0 and b=1 in (1.28)

$$\varphi(x) = \frac{1}{\pi i} \int_L^* \frac{f(t)}{t-x} dt. \tag{1.30}$$

For the unclosed contour the Poincare-Bertrand formula (1.11) is not applicable.

Then to solve such problems, supplement the contour L with other contour $L^{'}$, so as to form a close contour $L+L^{'}$ where $\varphi(t)=0$, $t \in L^{'}$, so that the interior and exterior of this closed contour stand for the positive and negative direction (Pogorzelski, 1966). Thus we have

$$\left.\begin{aligned} \Phi^{+}(x) &= \frac{1}{2}\varphi(x) + \frac{1}{2\pi i} \int_L^* \frac{\varphi(t)}{t-x} dt \\ \Phi^{-}(x) &= -\frac{1}{2}\varphi(x) + \frac{1}{2\pi i} \int_L^* \frac{\varphi(t)}{t-x} dt \end{aligned}\right\}, \tag{1.31}$$

which is called Sokhotski formulas (Gakhov, 1966).

These formulas can also be rewritten as

$$\left.\begin{aligned} \varphi(x) \quad &= \Phi^{+}(x) - \Phi^{-}(x) \\ \frac{1}{\pi i} \int_L^* \frac{\varphi(t)}{t-x} dt &= \Phi^{+}(x) + \Phi^{-}(x) \end{aligned}\right\} \tag{1.32}$$

By substituting (1.32) into equation (1.23) then the solution of (1.23) is reduced to solving the Riemann-Hilbert problem,

$$(a+b)\Phi^{+}(x) - (a-b)\Phi^{-}(x) = f(x) \tag{1.33}$$

where the solution will be of the form(Kanwal, 1997)