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On Nonspherical Partial Sums of Fourier Integrals of Continuous Functions from the Sobolev Spaces

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ABSTRACT

The partial integrals of the N-fold Fourier integrals connected with elliptic polynomials (not necessarily homogeneous; principal part of which has a strictly convex level surface) are considered. It is proved that if $a + s > (N - 1)/2$ and $ap = N$ then the Riesz means of the nonnegative order s of the N-fold Fourier integrals of continuous finite functions from the Sobolev spaces $W_p^a(R^N)$ converge uniformly on every compact set, and if $a + s > (N - 1)/2$ and $ap = N$, then for any $x_0 \in R^N$ there exists a continuous finite function from the Sobolev space such, that the corresponding Riesz means of the N-fold Fourier integrals diverge to infinity at x_0. AMS 2000 Mathematics Subject Classifications: Primary 42B08; Secondary 42C14.

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INTRODUCTION

Let Ω be an arbitrary domain in R^N, $\Omega \subseteq R^N$. Consider an arbitrary non-negative elliptic differential operator $A(x,D)$ with smooth coefficients and the domain of definition $C_0^\infty(\Omega)$ (i.e. infinitely differentiable functions on Ω with a compact support). Let us denote by A an arbitrary non-negative self-adjoint extension in $L_2(\Omega)$ of the operator $A(x, D)$. According to the spectral theorem there exists a family of spectral projectors $\{E_\lambda\}$ such that for every $f \in L_2(\Omega)$ one has

$$Af(x) = \int_0^\infty \lambda \, dE_\lambda \, f(x),$$

where $E_\lambda f(x)$ is called the eigenfunction expansions of f. According to Gärding theorem, each projector E_λ is an integral operator with the kernel $\theta(x, y, \lambda)$, called the spectral function (Gärding, 1954).

The Riesz means of nonnegative order s of eigenfunction expansions $E_\lambda f(x)$ are defined as

$$E_\lambda^s f(x) = \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s dE_t f(x) \tag{1}$$

In particular, if the operator $A(x, D)$ has constant coefficients and $\Omega = R^N$, then $E_\lambda f(x)$ coincides with the partial integrals of N-fold Fourier integrals of the function $f \in L_2(R^N)$.

In the present paper we study the uniform convergence of the spectral resolutions $E_\lambda^s f$ and their Riesz mean $E_\lambda^s f$ for functions belonging to the Sobolev space $W_p^a(\Omega)$ (for the definition and basic properties of the spaces $W_p^a(\Omega)$ see the monograph (Sobolev, 1963)).

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Definitive sufficient conditions for the uniform convergence on any compact set $K \subset \Omega$ of $E_\lambda^s f$ to the finite function (i.e. with compact support) f belonging to $W_p^a(\Omega)$ (even to the broader Nikol'skii classes $H_p^a(\Omega)$ $H_p^a(\Omega)$) were established by Sh.A. Alimov (1967; 1978), and they are as follows:

$$a + s \geq \frac{N-1}{2}, ap > N \tag{2}$$

where $p \geq 1, s \geq 0, a > 0$.

These conditions ensure the uniform convergence of eigenfunction expansions of Schrödinger operator with singular potentials too (Sh.A. Alimov and Joó, 1983).

The relations (2) were first found for the Laplace operator in the work of Il'in [5] for $s = 0$. Moreover, Il'in proved (*see* [3]), that for uniform convergence the first condition here is best possible. Namely, if $a + s < (N-1)/2$ then there exists a finite function $f \in C^a(\Omega)$ for which the means $E_\lambda^s f$ are unbounded at some point.

As for the condition $ap > N$, it guarantees, according to imbedding theorems, the function in question to be continuous, and if the opposite inequality $ap \leq N$ is satisfied, then there exists an unbounded finite function $f \in W_p^a(\Omega)$ whose Riesz means clearly cannot converge to it uniformly. In this connection the following question arises: is the assertion on uniform convergence valid if in conditions (2) the inequality $ap > N$ is replaced by the equality $ap = N$ and it is additionally required that the function f be continuous in the domain Ω? In the paper [6] Sh.A. Alimov gave a complete answer to this question in case of the Laplace operator. Namely, he proved that in case $ap = N$ the sum $a + s$ is essential, i.e. if $a + s < (N-1)/2$ then we have the uniform convergence, and if $a + s < (N-1)/2$ then we do not.

To prove these theorems Sh.A. Alimov, distinguishing the leading term of the spectral expansion (Il'in, 1957), obtained an extremely convenient for studying representation of $E_\lambda^s f$ for the functions f from $W_2^a(\Omega)$ (*see* Lemma 4). The proof of this representation essentially based on the mean value formula for the eigenfunctions of the Laplace operator. Even in case of the Shrödinger operator, where we have the mean value formula but with the remainder term, this representation is not proved yet.

In this paper we investigate the same question for elliptic differential operators (not necessarily homogeneous) with constant coefficients and an arbitrary order, considering in R^N.

THE MAIN RESULTS

Let $A(D)$ be an arbitrary elliptic differential expression with constant coefficients and order m:

$$A(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

Note this operator is not necessarily homogeneous, i.e. the coefficients a_α with $|\alpha| < m$ is not necessarily zero.

If we consider $A(D)$ on $L_2(R^N)$ with the domain of definition $C_0^\infty(R^N)$, i.e. infinitely differentiable functions with a compact support, then we will have essentially self-adjoint operator (see, for example (Alimov *et al*., 1991; Sobolev, 1963). Let us denote a unique self-adjoint extension of this operator by A. As mentioned above an eigenfunction expansion in this case coincides with the Fourier expansion and has the following form:

$$E_\lambda f(x) = (2\pi)^{-N/2} \int_{A(\xi) < \lambda} \hat{f} e^{ix\xi} d\xi$$

where $\hat{f}(\xi)$ is the Fourier transform of $f(x)$ and $A(\xi)$ is the symbol of the expression $A(D)$, i.e.

$$A(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha.$$

The Riesz means of a nonnegative order s of $E_\lambda f$ are defined as in (1). Observe if $s = 0$ then

$$E_\lambda^0 f(x) = E_\lambda f(x).$$

If we define

$$\theta^s(x, \lambda) = (2\pi)^{-N} \int_{A(\xi) < \lambda} \left(1 - \frac{A(\xi)}{\lambda}\right)^s e^{ix\xi} d\xi \tag{3}$$

then by the definition of the Fourier transform one has

$$E_\lambda^s f(x) = \int_{R^N} f(y) \theta^s(x - y, \lambda) dy. \tag{4}$$

The function $\theta(x, \lambda) = \theta^0(x, \lambda)$ is called the spectral function of the operator A while $\theta^s(x, \lambda)$ is called the Riesz means of the spectral function.

The following homogeneous polynomial $A_0(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$ is called a principal symbol of the elliptic polynomial $A(\xi)$. Obviously, the asymptotic behavior of the function $\theta^s(x, \lambda)$ essentially depends on the geometry of the set $C = \{\xi \in R^N : A_0(\xi) \leq 1\}$. When the surface

$$\partial C = \{\xi \in R^N : A_0(\xi) = 1\} \tag{5}$$

is strictly convex, i.e. when the Gaussian curvature is positive at every point of this surface, then we have the best possible estimate for the Riesz means of the spectral function.

We denote by $W_{0,p}^a(R^N)$ the class of functions belonging to $W_p^a(R^N)$ and having compact support (i.e. finite functions).

The main results of the paper are the following theorems.

Theorem 1. *Let* $A(\xi)$ *be an arbitrary elliptic polynomial with strictly convex surface* (5). *Let* $G \subset R^N$ *and suppose that the numbers* $s \geq 0$, $p \geq 1$ *and the integer* $a > 0$ *are related by*

$$a + s > \frac{N-1}{2}, ap = N.$$

Then for any function $f \in W_{0,p}^a(R^N)$ *continuous in the domain G the following equality holds uniformly on each compact set* $K \subset G$:

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x).$$

Theorem 2. *Let* $A(\xi)$ *be an arbitrary elliptic polynomial with a strictly convex surface* (5), *and let* x_0 *be an arbitrary point of* R^N. *Suppose that the numbers* $s \geq 0$, $p \geq 1$, *and the integer* $a > 0$ *are related by*

$$a + s = \frac{N-1}{2}, ap = N.$$

Then there exists a continuous function $f \in W^a_{0,p}(R^N)$ *such that*

$$\overline{\lim_{\lambda\to\infty}} E^s_\lambda f(x_0) = +\infty.$$

These theorems are true in fact for finite functions from the broader Nikol'skii classes $H^a_p(R^N)$ where the index $a > 0$ may assume any (not necessarily integer) values. But for these functions the proofs will be technically more complicated.

We also note, that these theorems were proved in case of homogeneous elliptic operators $A_0(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ (Alimov *et al.*, 1991). In case of the operators $A_0(D)$ the sets $\{\xi \in R^N: A_0(\xi) \le \lambda\}$ form a family of enclosing sets, whereas the domains $\{\xi \in R^N: A(\xi) \le \lambda\}$ are in general not similar for different values of λ. Therefore, investigation of the Riesz means (4) for non-homogeneous elliptic operators more complicated.

CONCLUSION

Let $A(D)$ be a non-homogeneous elliptic operator with strictly convex surface (5). We investigated the corresponding Riesz means of the Fourier integrals (4). Consider the Sobolev class $W^a_p(R^N)$ and let $ap = N$. In this case functions from this class are not necessarily continuous. Therefore we cannot expect the uniform convergence of multiple Fourier integrals to these functions. Now let us consider only those functions of $W^a_p(R^N)$ which are continuous (i.e. consider a subspace). As it is shown in Theorems 1 and 2, for uniform convergence of the Riesz means (4) of order s for functions from this subspace the sum $a + s$ is essential. In other words, if this sum is greater than $(N - 1)/2$ then we have uniform convergence and otherwise we do not.

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