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The Behaviour of the Solutions of the Forced Van der Pol Equation with Small Parameters

MAT ROFA BIN ISMAIL
Department of Mathematics,
Faculty of Science and Environmental Studies,
Universiti Pertanian Malaysia,
43400 Serdang, Selangor, Malaysia.

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ABSTRAKS

Persamaan Van der Pol dengan suatu daya $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = p(t), \mu > 0, |p(t)| \leqslant k$ *iaitu* μ *dan* k *merupakan parameter-parameter kecil, yang diselidiki dalam satah fasa. Akan ditunjukkan bahawa penyelesaian persamaan tersebut akan beransur-ansur terkepung dalam suatu anulus apabila t bertambah untuk nilai-nilai tertentu bagi* μ *dan k.*

ABSTRACT

The forced Van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = p(t), \ \mu > 0, | p(t)| \leqslant k$ *where* μ *and* k *are small parameters, is investigated in the phase plane, It is shown that the solutions of the equation are eventually confined in an annulas as t increases for certain values of* μ *and k.*

1. INTRODUCTION

The differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = (p(t), \mu > 0 \quad (1.1)$$

is known as the forced Van der Pol equation. It is shown that the unforced equation of (1.1) possesses an isolated periodic solution which is known as a limit cycle, (Jordan and Smith, 1977). For μ small, the limit cycle approximates a circle of radius 2, in the phase plane.

The periodical forced Van der Pol equation is particularly interesting because it represents a system with autonomous oscillations influenced by a second external cyclic force. This phenomenom, with μ large, was investigated by Cartwright and Littlewood (1945) and Levinson (1949). In this case, due to the existence of two competing factors (the damping and forcing terms) the trajectories of the equation may wander aimlessly before settling down in a particular domain. In more recent terminology, this would be called transient chaos. The discussion of this matter can be found in Guckenheimer and Holmes (1983) and Thompson and Steward (1986).

The case in which $p(t)$ is a continous and bounded function was investigated by James (1974) in connection with control theory with particular attention to swiching curves, while Ponzo (1967) considered the more general equation of the form

$$\frac{d^2x}{dt^2} + \mu f(x)\frac{dx}{dt} + g(x) = \mu p(t),$$

for μ large. The latter shows that under suitable restriction of f, the solution will eventually be confined in a region.

In this paper, we try to investigate the behaviour of the solutions of the equation (1.1) when $p(t)$ is continuous and bounded by k, that is, $|p(t)| \leqslant k$ for all t. We will show that for μ and k small, all solutions of the equation are bounded in a domain containing the origin or in an annulus encircling the origin depending on the values of μ and k.

2. THE PHASE PLANE

In the phase plane, equation (1.1) is equivalent to the system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\mu(x^2 - 1)y - x + p(t), \ p(t) \leqslant k.$$

Now, consider the system,

$$\frac{dx}{dt} = y \qquad (2.2)$$

$$\frac{dy}{dt} = -\mu(x^2 - 1)y - x + k$$

and the system

$$\frac{dx}{dt} = y \qquad (2.3)$$

$$\frac{dy}{dt} = -\mu(x^2 - 1)y - x - k$$

For convenience, we denote systems (2.1), (2.2) and (2.3) by S, S+ and S– respectively. Since

$$-k \leqslant p(t) \leqslant k,$$

then it follows that

$$\frac{-\mu(x^2-1)y-x\ k}{y} \leqslant \frac{\mu(x^2-1)y-x+p(t)}{y} \leqslant \frac{-\mu(x^2-1)\ -x+k}{y} \qquad (2.4)$$

for $y > 0$; that is, the slope of the trajectory of S in the upper half plane of the phase is greater than the slope of the trajectory of S_- but less than the slope of the trajectory of S_+ at a particular point. One can see that, the trajectory of S_+ in $y > 0$ and the trajectory of S_- in $y < 0$ are symmetric about the origin.

3. THE OUTER AND INNER BOUNDARIES

We define an outer boundary of the solutions of equation (2.1) as a closed curve which has the property that at every point on the curve, the vector field of the solutions of S point towards its interior. Similarly, we define an inner boundary of the solution of equation (2.1) as a closed curve which has the property that at every point on it the vector field of the solutions of S point outwards from its interior.

It can be shown that if one chooses the trajectory of S_+ in $y > 0$ and trajectory of S_- in $y < 0$, that is, if one choses the system

$$\begin{array}{ll} S_+ \text{ when } y > 0 \\ S_- \text{ when } y < 0 \end{array} \qquad (3.1)$$

then this system will serve as an outer boundary of the trajectories of S if the above systems form a closed curve (or a limit cycle). (In the phase plane, a periodic solution is represented by a closed curve). In contrast, if one chooses the system

$$\begin{array}{ll} S_- \text{ when } y > 0 \\ S_+ \text{ when } y < 0 \end{array} \qquad (3.2)$$

and if this system has a closed curve then it will serve as an inner boundary for the trajectories of S.

Now, the problem of confining the solutions of equation (2.1) is coverted to the problem of the existence of limit cycles of the systems (3.1) and (3.2).

Since μ and k are assumed to be small, the solutions of system S_+ and S_- will be approximated to the solutions of the equations

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x$$

whose solutions are a family of circles centred at the origin, which can be expressed as a pair of parametric equations,

$$\begin{array}{l} x(t) = a \cos t \\ y(t) = -a \sin t \end{array} \qquad (3.3)$$

where a is a positive constant be to determined. Similarly, if system (3.2) has a limit cycle then it will be approximated to the form of (3.3). Since μ and k are small in system (3.1), we shall define the energy involved in the system as

$$E(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 \;;$$

and we consider the change in the energy along the path of solutions of equations (3.1),

$$\frac{dE}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$= \begin{cases} y[x - \mu(x^2 - 1)y - x + k] & y > 0 \\ y[x - \mu(x^2 - 1)y - x - k] & y < 0 \end{cases}$$

$$= \begin{cases} y(t)[-\mu(x^2(t) - 1)y(t) + k], & t \in (\pi, 2\pi) \\ y(t)[-\mu(x^2(t) - 1)y(t) - k], & t \in (0, \Pi) \end{cases} \tag{3.4}$$

where $x(t) \sim -a \cos t$, $y(t) \sim -a \sin t$. Integrating equations (3.4) with repect to t from 0 to 2^{π}, we have

$$E(2\pi) - E(0) = -\frac{\mu a^4 K}{4} + \mu a^2 \pi + 4ak.$$

Around a limit cycle, we require $E(2\pi) - E(0) = 0$

$$\frac{\mu a^4 \pi - \mu a^2 \pi - 4ak}{4} = 0.$$

This implies that $a = 0$ (which represents the approximation to the critical point), or

$$a^3 - 4a - \frac{16k}{\pi\mu} = 0 \tag{3.5}$$

It is easily shown that equation (3.5) has one positive real root for all μ and k positive. Let this root be a_o.

$$L_o : x(t) = a_o \cos t$$
$$y(x) = -a_o \sin t \tag{3.6}$$

is an outer boundary for the solutions of S. Also it can be shown that L_o is stable, in the sense that the energy is gained for $a < a_o$ and is lost for $a > a_o$.

For the inner boundary, we again consider the change of the energy along the path of solutions of systems (3.2),

$$\frac{dE}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt},$$

$$= \begin{cases} y[-\mu(x^2 - 1)y - k], & y > 0 \\ y[-\mu(x^2 - 1)y + k], & y < 0, \end{cases}$$

$$= \begin{cases} y(t)[-\mu(x^2(t) - 1)y(t) - k], & t \in (\pi, 2\pi) \\ y(t)[-\mu(x^2(t) - 1)y(t) + k], & t \in (0, \pi) \end{cases}$$

where $x(t) \sim a$ cost, $y(t) \sim a$ sint. Integrating the above equations with respect to t from 0 to 2π, we have

$$E(2\pi) - E(0) = -\frac{\mu a^4 \pi}{4} + \mu a^2 \pi - 4ak$$

Around a limit cycle we need $E(2\pi) - E(0) = 0$, that is,

$$-\frac{\mu a^4 \pi}{4} + \mu a^2 \pi - 4ak = 0 \tag{3.7}$$

so $a = 0$ (represents the approxination of the critical point), or

$$a^3 - 4a + \frac{16k}{\mu\pi} = 0 \tag{3.8}$$

It can be shown that equation (3.8) has either two positive real roots, one positive real root or no positive real root according to

$$\frac{k}{\mu} < \frac{\pi}{3\sqrt{3}}, \quad \frac{k}{\mu} = \frac{\pi}{3\sqrt{3}} \quad \text{or} \quad \frac{k}{\mu} > \frac{\pi}{3\sqrt{3}}$$

respectively.

In the case $\frac{k}{\mu} < \frac{\pi}{3\sqrt{3}}$, equation (3.8) has two real roots, say a_1 and a_2 with $a_1 < a_2$. It can be easily shown that $a_1 < \frac{2}{\sqrt{3}} < a_2$. Comparing with a_o, we have

$$0 < a_1 < \frac{2}{\sqrt{3}} < a_2 < 2 < a_o.$$

(Can be seen by a geometrical argument).

So

$$L_1: \begin{aligned} x(t) &= a_1 \cos t \\ y(t) &= -a_1 \sin t \end{aligned} \qquad (3.9)$$

and

$$L_2: \begin{aligned} x(t) &= a_2 \cos t \\ y(t) &= -a_2 \sin t \end{aligned} \qquad (3.10)$$

are inner boundaries to the solutions of S. By using an energy argument, it can be shown that the limit cycle L_1 is unstable but the limit cycle L_2 is stable.

4. CONCLUSION

(i) The limit cycle L_o, in fact, exists for all small values of μ and k, and it confines the solutions of S in its interior, as it is an outer boundary to the solutions (by the defination).

(ii) When $\frac{k}{\mu} > \frac{\pi}{3\sqrt{3}}$, there is no inner boundary for the solutions of S (but the outer boudary L_o always exists for all cases). One the trajectories of S cross the outer boundary from outside they stay permanently in the interior of L_o; *(Figure 1)*.

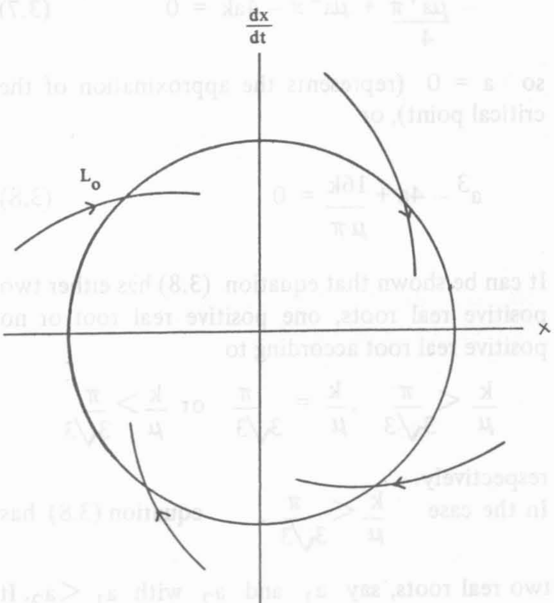

Fig 1: The limit cycle L_o serves as an outer boundary for the trajectories.

(iii) When $\frac{k}{\mu} < \frac{\pi}{3\sqrt{3}}$, system (3.2) has two limit cycles, namely L_1 and L_2. These limit cycles are inner boundaries to the solutions of S. We have shown that L_1 is enclosed by L_2 (by the fact that $a_1 < a_2$) and L_2 is enclosed by L_o (see *Figure 2*). As a result, the solutions of S eventually enter the region between L_2 and L_o as t increase. Now we consider a solution which crosses the limit cycle L_1 from its interior. The solution then eventually crosses L_2 to stay permanently in the annulus between L_2 and L_o, by the fact that L_1 is unstable, so all the trajectories of S_- exterior to this limit cycle converge to the stable limit cycle L_2. As a result the solution which crosses the inner boundary L_1 (because its slope is greater than the slope of the trajectory of S_- at every point eventually crosses the limit cycle L_2, and stays permanently in the annulus.

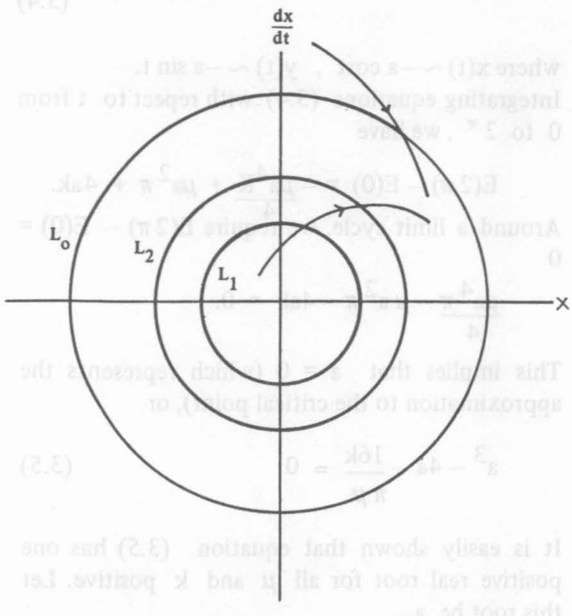

Fig 2: The confinement of the trajectories in the annulus between L_2 and L_o.

(iv) The case when $\frac{k}{\mu} = \frac{\pi}{3\sqrt{3}}$ is similar, except that L_1 and L_2 are coincident and the solutions of S stay permenantly in the annulus for all $t > t_o$, where t_o is the time when the trajectories cross the inner and outer boundaries simultaneously.

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