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Automorphisms of Fuchsian Groups of Genus Zero

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ABSTRAK

Setiap automorfisma kumpulan Fuchsan diaruh oleh suatu automorfisma kumpulan bebas. Kertas ini memberikan suatu persembahan kumpulan automorfisma bagi kumpulan Fuchsan genus sifar melalui kumpulan tocang. Sebagai sampingannya kumpulan kelas pemetaan tulen dan kumpulan serabut Seifert dibincangkan.

ABSTRACT

Every automorphism in Fuchsian group is induced by some automorphism of a free group. This paper gives a presentation of a automorphism group of Fuchsian group of genus zero via braid groups. We also obtained the pure mapping class groups and the Seifert Fibre Groups.

INTRODUCTION

A *co-compact Fuchsian group*, Γ, is known to have the following presentation:

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, x_1, x_2, \ldots, x_r \mid x_i^{m_i} = \prod_{i=i}^{r} x_i \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle$$

where $g \geqslant 0$, $r \geqslant 0$, $m_i \geqslant 2$ and $[a, b] = aba^{-1}b^{-1}$. (See [5]). The integers $m_1, m_2, \ldots, m_r$ are called the *periods* and g is called the *genus*. We say Γ has *signature* $(g; m_1, m_2, \ldots, m_r)$. If $g = 0$, we simply write $(m_1, m_2, \ldots, m_r)$ for $(0; m_1, m_2, \ldots, m_r)$. If $g = 0$, $r = 3$, we call (ℓ, m, n) the *triangle group*.

Γ is the fundamental group of some surface. By Nielsen's theorem, every automorphism in the fundamental group of a surface is induced by a self-homeomorphism of the surface. With abuse of language, we call those automorphisms induced by the orientation-preserving self-homeomorphisms of the surface, the orientation-preserving automorphisms, denoted by Aut^+. In this paper, we will give a presentation of $\mathrm{Aut}^+ \Gamma$, for Γ a Fuchsian group of genus zero.

1. BRAID GROUPS

Artin (1925, 1947) defined the *braid group* (the full braid group) of the plane, Br, with r strings as:

Generators: σ_i , $1 \leqslant i \leqslant r-1$.

Defining relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leqslant i \leqslant r-2 \tag{1.1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geqslant 2$$

The braid group, B_r, can be looked upon as the subgroup of the automorphism group of a free group of rank r. We will adopt the convention of operating from right to left, that is

$$\sigma_i \sigma_j(x) = \sigma_i(\sigma_j(x)).$$

Let $\upsilon\colon B_r \to \Sigma_r$ be defined by $\upsilon(\sigma_i) = (i\ i+1)$, for $1 \leqslant i \leqslant r-1$, where Σ_r is a symmetric group on r letters. Let $P_r = \ker \upsilon$. Then P_r is called the *pure braid group* and is known to have the following presentation: Generators:

$$A_{ij} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i^{2}\sigma_{i+1}\ldots\sigma_{j-2}\sigma_{j-1},\ 1 \leqslant i < j < r$$

Defining relations:

$$A_{st}^{-1} A_{ij} A_{st}$$

$$= A_{ij}, \text{ if } s < t < i < j \text{ or } i < s < t < j$$

$$= A_{sj}^{-1} A_{ij} A_{sj}, \text{ if } t = i$$

$$= A_{ij}^{-1} A_{tj}^{-1} A_{ij} A_{tj} A_{ij}, \text{ if } s = i < j < t \quad (1.2)$$

$$= A_{sj}^{-1} A_{tj}^{-1} A_{sj} A_{tj} A_{ij} A_{tj}^{-1} A_{sj}^{-1} A_{tj} A_{sj}, \quad \text{if } s < i < t < j$$

As a representation of the automorphism of the free group $F_r = \langle x_1, x_2, \ldots, x_r \rangle$, we have.

$$\begin{aligned} c_i : x_i &\to x_i x_{i+1} x_i^{-1} \\ x_{i+1} &\to x_i && (1.3) \\ x_j &\to x_j, && \text{for } j \neq i, i+1. \end{aligned}$$

and

$$\begin{aligned} A_{st} : x_i &\to x_i && \text{, if } t < i \text{ or } i < s \\ &\to x_s x_i x_s^{-1} && \text{, if } t = i \\ &\to x_i x_t x_i x_t^{-1} x_i^{-1} && \text{, if } s = i \qquad (1.4) \\ &\to x_s x_t x_s^{-1} x_t^{-1} x_i x_t x_s x_t^{-1} x_s^{-1} && \text{, if } s < i < t \end{aligned}$$

Note:

$$(\sigma_1\sigma_2 \ldots \sigma_{r-1})^r = (\sigma_{r-1}\sigma_{r-2}\ldots\sigma_1)^r$$

$$= I(x_1 x_2 \ldots x_r) \quad (1.5)$$

$$(A_{r-1,r}A_{r-2,r} \ldots A_{2r}A_{1r})(A_{r-2,r-1}A_{r-3,r-1} \ldots (A_{23}A_{13})(A_{12}) \ldots A_{1,r-1}) \ldots \quad (1.6)$$

$$= (A_{12})(A_{23}A_{13}) \ldots (A_{r-1,r}A_{r-2,r} \ldots A_{1r})$$

$$= I(x_1 x_2 \ldots x_r).$$

where $I(\gamma)$ denotes the inner automorphism

$$x \to \gamma x \gamma^{-1}$$

The center of $B_r, r \geqslant 3$, is the infinite cyclic subgroup generated by

$$a^r = (\sigma_1\sigma_2\ldots\sigma_{r-1})^r = (A_{12})(A_{23}A_{13}) \ldots (A_{r-1,r}A_{r-2,r} \ldots A_{1r}).$$

$$\ldots \quad (1.7)$$

(See Birman, 1974 and Chow, 1948)

We now state the well-known necessary and sufficient condition for an automorphism of a free group to be an element of the braid group B_r.

Theorem 1

Let $F_r = \langle x_1, x_2, \ldots, x_r \rangle$. Then $\beta \in B_r \subset \text{Aut } F_r$ if and only if β statisties:

$$\beta(x_i) = \lambda_i x_{\mu i} \lambda_i^{-1} \quad . \quad 1 \leqslant i \leqslant r$$

$$\beta(x_1 x_2 \ldots x_r) = x_1 x_2 \ldots x_r$$

where $\begin{pmatrix} 1 & 2 & \ldots & r \\ \mu_1 & \mu_2 & \ldots & \mu_r \end{pmatrix}$ is a permutation and

$$\lambda_i = \lambda_i (x_1, x_2, \ldots, x_r).$$

(Artin, 1925 and Birman, 1974) (See [1], [3])

The mapping class groups are closely related to the braid groups and the automorphism groups of the Fuchsian groups. (See [3], [7]). The mapping class group (full mapping class group), (M(o, r), is known to have the following presentation:

Generators: ξ_i , $1 \leqslant i \leqslant r-1$

Defining relations:

$$\xi_i \xi_{i+1} \xi_i = \xi_{i+1} \xi_i \xi_{i+1} \quad , 1 \leqslant i \leqslant r-2$$

$$\xi_i \xi_j = \xi_j \xi_i \quad , |i - j| \geqslant 2 \qquad (1.8)$$

$$\xi_1 \xi_2 \ldots \xi_{r-2} \xi_{r-1}^2 \xi_{r-2} \ldots \xi_2 \xi_1 = 1$$

$$(\xi_1 \xi_2 \ldots \xi_{r-1})^r = 1$$

2. AUTOMORPHISM GROUPS

We now state a restricted version of Zieschang's theorem (1966):

Theorem 2.1.

Let $\Gamma = < x_1, x_2, \ldots, \ldots, x_r \,[x_i^{m_i} = x_1 x_2 \ldots x_r = 1 >$ be a Fuchsian group of genus zero and $\widehat{\Gamma} = < \widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_r >$ be a free group of rank r. Then every $\phi \in \mathrm{Aut}^+ \Gamma$ is induced by some $\widehat{\phi} \in \mathrm{Aut}\, \widehat{\Gamma}$ staisfying:

$$\begin{aligned} \widehat{\phi}(\widehat{x}_i) &= \widehat{\lambda}_i \widehat{x}_{\mu_i} \widehat{\lambda}_i^{-1}, \quad 1 \leqslant i \leqslant r \\ \widehat{\phi}(\widehat{x}_1 \widehat{x}_2 \ldots \widehat{x}_r) &= \widehat{\lambda}(\widehat{x}_1 \widehat{x}_2 \ldots \widehat{x}_r)\widehat{\lambda}^{-1} \end{aligned} \qquad (2.1.)$$

where $\begin{pmatrix} 1 & 2 & \ldots & r \\ \mu_1 & \mu_2 & \ldots & \mu_r \end{pmatrix}$ is a permutation with $m_{\mu_i} = m_i$, $1 \leqslant i \leqslant r$

and $\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_r, \widehat{\lambda} \in \widehat{\Gamma}$

Let $\psi: \widehat{\Gamma} \to \Gamma$, $\psi(\widehat{x}_i) = x_i$, $1 \leqslant i \leqslant r$, be the natural homomorphism. If $\widehat{\phi} \in \mathrm{Aut}\, \widehat{\Gamma}$ satifies (2.1.), then there is a unique $\phi \in \mathrm{Aut}^+ \Gamma$ defined by:

$$\begin{array}{ccc} \widehat{\Gamma} & \xrightarrow{\ \widehat{\phi}\ } & \widehat{\Gamma} \\ \downarrow \psi & & \downarrow \psi \\ \Gamma & \xrightarrow{\ \phi\ } & \Gamma \end{array} \qquad (2.2)$$

We know that every automorphism of Γ can be obtained in this way by Theorem 2.1. The set of all such automorphisms $\widehat{\phi}$ of $\widehat{\Gamma}$ forms a subgroup of Aut $\widehat{\Gamma}$ which is denoted by $\widehat{A(\Gamma)}$. By definition, $B_r \subset \widehat{A(\Gamma)}$. The correspondence $\widehat{\phi} \to \phi$ defines a homomorphism η: $\widehat{A(\Gamma)} \to \mathrm{Aut}^+ \Gamma$. We denote $\eta(B_r) = B_r^*$, $\eta(P_r) = P_r^*$. Without ambiguity, we will use the same symbol for the elements in B_r^*, (respectively, P_r^*), corresponding to the elements in B_r, (respectively, P_r).

As we see, $\phi \in \mathrm{Aut}^+ \Gamma$ maps x_i into a conjugate of $x_{\mu i}$ with $m_{\mu i} = m_i$. The intermediate groups between P_r^* and B_r^* (and hence the intermediate groups between P_r and B_r) depend strongly on the periods and the permutation. Let $\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i} = \Sigma_{\alpha_1} \times \ldots \times \Sigma_{\alpha_k}$ where $\sum_{i=1}^{k} \alpha_i = r$, be the symmetric group corresponding to the permutation of the periods. Then we have:

$$\upsilon : B_r \to \Sigma_r \supset \overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i} \supset \{1\}$$

We are interested in the structure of the groups $\upsilon^{-1}(\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i})$ and $\eta\upsilon^{-1}(\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i})$ defined by:

$$\begin{array}{ccccc} B_r^* & \supset & \eta\upsilon^{-1}(\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i}) & \supset & P_r^* \\ \uparrow \eta & & \uparrow \eta & & \uparrow \eta \\ B_r & & \upsilon^{-1}(\overset{k}{\underset{i=1}{\Pi}} \Sigma_{\alpha_i}) & & P_r \\ \downarrow \upsilon & & \downarrow \upsilon & & \downarrow \upsilon \\ \Sigma_r & \supset & \overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i} & \supset & [1] \end{array} \qquad (2.3)$$

Let us simplify the notation of the signature of Γ as:

$$(m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_k^{\alpha_k}) \qquad (2.4)$$

where $\sum_{i=1}^{k} \alpha_i = r$, to mean that the first α_1 generators have period m_1, the next α_2 generators have period m_2, ..., and the last α_k generators have period m_k. We set $\alpha_o = 0$, the significance of which will become clear later for the simplicity of notation. Let $\ell_n = \sum_{i=0}^{n} \alpha_i$, $0 \leqslant n \leqslant k$. Then $\ell_0 = 0, \ell_1 = \alpha_1, \ell_2 = \alpha_1 + \alpha_2, \ldots, \ell_k = r$.

Then the defining relations of Γ with signature (2.4) are:

$$x_1 x_2 \ldots x_r = 1$$

$$x_i^{m_{n+1}} = 1, \text{ for } \ell_n + 1 \leqslant i \leqslant \ell_{n+1}, 0 \leqslant n \leqslant k-1.$$

From the homomorphism υ, we then see that the generators of $\upsilon^{-1}(\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i})$ are:

σ_i for $1 \leq i \leq r-1, i \neq \ell_n, 1 \leq n \leq k-1$. (2.6)

A_{ij} for $1 \leq i < j \leq r$. (2.7)

From the definition of A_{ij} in terms of σ_i's, we see that it suffices to substitute (2.7) with:

$A_{i,\ell_n} + 1$ for $1 \leq i \leq \ell_n, 1 \leq n \leq k-1$. (2.8)

Hence, (2.6) and (2.8) form a sufficient set of generators of $\upsilon^{-1}(\prod_{i=1}^{k} \Sigma_{\alpha_i})$.

The defining relations are those of the braid group and the pure braid group wherever definable corresponding to the symmetric group $\pi_{i=1}^{k} \Sigma_{\alpha_i}$. We then have the following:

Theorem 2.2.

$\upsilon^{-1}(\pi_{i=1}^{k} \psi_{\alpha_i})$ admits a presentation with generators:

σ_i, for $1 \leq i \leq r-1, i \neq \ell_n, 1 \leq n \leq k-1$

A_{i,ℓ_n+1}, for $1 \leq i \leq \ell_n, 1 \leq n \leq k-1$

and defining relations: (1.2)

and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $1 \leq i \leq r-1$, $i \neq \ell_n - 1, \ell_n$

$\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i-j| \geq 2$

$A_{ij} \sigma_t = \sigma_t A_{ij}$, for $t \neq i-1, i, j$.

Theorem 2.3

Let Γ be a Fuchsian group with signature (2.4). Then.

$$\mathrm{Aut}^+ \Gamma = \eta \upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i}).$$

Lemma. 2.1.

$$I(\Gamma) \subset P_r^* \subset \eta \upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i}) \subset B_r^*$$

Proof:

If we denote the inner automorphisms

$$x_j \to (x_1 x_2 \ldots x_\ell) x_j (x_1 x_2 \ldots x_\ell)^{-1}, 1 \leq j \leq r$$

by θ_ℓ, $1 \leq \ell \leq r$, then we have the following:

$$\theta_1 = (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_2)^{1-r}$$

$$\theta_i = (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_{i+1})^{i-r} (\sigma_{i-1} \sigma_{i-2} \ldots \sigma_1)^i \quad (2.9)$$

$$\theta_{r-1} = (\sigma_{r-2} \sigma_{r-3} \ldots \sigma_2 \sigma_1)^{r-1}$$

$$\theta_r = (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_1)^r = (\sigma_1 \sigma_2 \ldots \sigma_{r-1})^r$$

where σ_i's now are the elements of B_r^*. Since each element x_i is mapped on a conjugate, it follows then by definition that $I(\Gamma) \subset P_r^*$.

Remarks 2.1.

1. Note that with the action on Γ (that is, considering σ_i's as the elements of B_r^*)

$$\sigma_1 \sigma_2 \ldots \sigma_{r-2} \sigma_{r-1}^2 \sigma_{r-2} \ldots \sigma_2 \sigma_1$$

$$= (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_3 \sigma_2)^{1-r} (\sigma_{r-1} \ldots \sigma_2 \sigma_1)^r$$

$$= (\sigma_{r-1} \sigma_{r-2} \ldots \sigma_2)^{1-r}$$

II. If x_i and x_j have equal periods, then their inner automorphisms are conjugate of each other; Since the periods are equal, there is an automorphism

$$\gamma : x_i \to x_j$$

such that for each k, $1 < k < r$,

$$[\gamma I(x_i) \gamma^{-1}](x_k) = \gamma I(x_i)(\gamma^{-1}(x_k))$$

$$= \gamma(x_i \gamma^{-1}(x_k) x_i^{-1})$$

$$= \gamma(x_i) x_k \gamma(x_i^{-1}) = x_j x_k x_j^{-1}$$

$$= [I(x_j)](x_k).$$

Therefore, $I(x_j) = \gamma I(x_i) \gamma^{-1}$.

Proof of Theorem 2.3.

By Zieschang's theorem, every $\phi \in \mathrm{Aut}^+ \Gamma$ is induced by $\widehat{\phi} \in \widehat{A(\Gamma)}$ which satisfies (2.1.). Then $\widehat{A(\Gamma)} = I(\Gamma).\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ and $\mathrm{Aut}^+ \Gamma = I(\Gamma).\eta\upsilon^{-1}$

$(\pi_{i=1}^{k} \Sigma_{\alpha_i})$. By Lemma 2.1. then we have the result.

Corollary: 2.1.

I. If all the periods are equal, then $Aut^+ \Gamma = B_r^*$.

II. If all the periods are distinct, then $Aut^+ \Gamma = P_r^*$

Our aim now is to find the structure of these groups $n\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$, B_r^*, P_r^*. We will do this in two stages.

Stage 1:

Let N_π be the normal closure of $\{\widehat{x}_1\widehat{x}_2...\widehat{x}_r\}$ in $\widehat{\Gamma}$ and $\Gamma_\pi = \widehat{\Gamma}/N_\pi$. Let $\eta_1\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ be the group of automorphisms in Γ_π induced by $\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$. Correspondingly, let $\eta_1(B_r) = B_r^1$ and $\eta_1(P_r) = P_r^1$.

Then by Magnus's theorem, (Maclachlan, 1973 and Magnus 1934), ker η_1 = center. Hence we have:

Theorem 2.4.

$\eta_1\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ is isomorphic to $\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ modulo the center.

Hence we can find the presentation of $\eta_1\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ by expressing $I(\widehat{x}_1\widehat{x}_2 ... \widehat{x}_r)$, which is the generator of the center by (1.7), in terms of the generators $\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_1})$.

Corollary 2.2.

$B_r^1 \cong B_r$/center. Therefore B_r^1 is generated by σ_i, $1 \leqslant i \leqslant r-1$, with defining relations (1.1) and

$$(\sigma_1\sigma_2 \ldots \sigma_{r-1})^r = 1$$

Corollary 2.3.

$P_r^1 \cong P_{r/center}$. Therefore P_r^1 is generated by A_{ij}, $1 \leqslant i < j \leqslant r$, with defining relations (1.2) and

$$(A_{12})(A_{23}A_{13}) \ldots (A_{r-1,r}A_{r-2,r} \ldots A_{1r}) = 1.$$

Remarks 2.2.

Maclachlan, (1973), gives the presentation of B_r^1. By the same argument as Theorem 2.3., $Aut^+ \Gamma_\pi = B_r^1$.

Stage 2:

Let Ω: $Aut^+ \Gamma_\pi \to Aut^+ \Gamma$ be the natural homomorphism with $\Omega(B_r^1) = B_r^*$, $\Omega(P_r^1) = P_r^*$, $\Omega(\eta_1\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})) = \eta\upsilon_1)) = \eta\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$. We will first find B_r^*.

Let K be the normal closure of $[I(x_i^{m}): 1 \leqslant i \leqslant r)$in B_r^1. We will now prove the following:

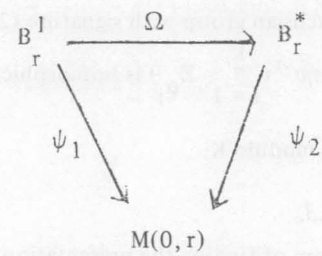
Theorem 2.5.

$$B_r^* \cong B_r^1/K.$$

Proof:

By Maclachlan & Harvey (1975) we have:

$$B_r^1/I(\Gamma_\pi) \cong Aut^+ \Gamma_\pi)/I(\Gamma_\pi) \cong M(0,r) \cong Aut^+\Gamma/I(\Gamma) \cong B_r^*/I(\Gamma).$$



with ker $\psi_1 = I(\Gamma_\pi)$, ker $\psi_2 = I(\Gamma)$. So, Ω^{-1} (ker ψ_2) $= I(\Gamma_\pi)$. Therefore ker $\Omega \subset I(\Gamma_\pi)$. Hence, ker $\Omega \subset K$. Clearly, $K \subset$ ker Ω. Thus ker $\Omega = K$ proving our theorem.

The extra relations that we have to add in B_r^* are those of $\{I(x_i^{m}); 1 \leqslant i \leqslant r\}$ By remark 2.1, it suffices to add only:

$$I(x_1^{m}) = (\sigma_1\sigma_2 \cdots \sigma_{r-2}\sigma_{r-1}^2\sigma_{r-2} \cdots \sigma_2\sigma_1)^m = 1.$$

Hence we have shown.

Theorem 2.6

If Γ is a Fuchsian group of genus zero with r equal periods, m, then $Aut^+ \Gamma$ is generated by σ_i, $1 \leqslant i \leqslant r-1$, with defining relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad , 1 \leqslant i \leqslant r-2$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad , |i-j| \geqslant 2$$

$$(\sigma_1\sigma_2 \cdots \sigma_{r-1})^r = 1$$

$$(\sigma_1\sigma_2 \cdots \sigma_{r-2}\sigma^2_{r-1}\sigma_{r-2} \cdots \sigma_2\sigma_1)^m = 1$$

We will next find $\eta\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$. Let K now be the normal closure of $\{I(x_k^{m_{n+1}}); \sigma \leqslant \eta \leqslant k-1, \ell_n + 1 \leqslant i \leqslant \ell_{n+1}\}$ in Γ_π. By a similar argument to (2.5), with the 'mapping class group' corresponding to $\pi_{i=1}^{k} \Sigma_{\alpha_i}$, then ker Ω = K. Hence we have the following.

Theorem 2.7.

If Γ is a Fuchsian group with signature (2.4.), then

$Aut^+ \Gamma = \eta\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ is isomorphic to $\eta_1\upsilon^{-1}$ $(\pi_{i=1}^{k} \Sigma_{\alpha_i})$ modulo K.

Remark 2.3.

Our problem of finding the presentation is reduced to expressing $\{I(x_i^{m_{n+1}}): 0 \leqslant n \leqslant k-1, \ell_n + 1 \leqslant i \leqslant \ell_{n+1}\}$ in terms of the generators of $n_1\upsilon^{-1}(\pi_{i=1}^{k} \Sigma_{\alpha_i})$, which depend on the signature of Γ.

Corollary 2.4.

If Γ is a Fuchsian group of genus zero and all the periods are distinct, then $Aut^+ \Gamma = P_r^*$ is isomorphic to P_r^1 modulo K, where K is the normal closure of $\{I(x_i^{m_i}): 1 \leqslant i \leqslant r, m_i \neq m_j \text{ for all } i \neq j\}$.

Examples

1. $\Gamma = \langle x_1, x_2, x_3, x_4 \mid x_1x_2x_3x_4 = x_i^{m_i} = 1$

$1 \leqslant i \leqslant 4, m_i \neq m_j \text{ for } i \neq j \rangle$

$Aut^+ \Gamma$ is generated by A_{ij}, $1 \leqslant i < j \leqslant 4$, with defining relations (1.2) and

$$A_{12}A_{23}A_{13}A_{34}A_{24}A_{14} = 1$$

$$(A_{34}A_{24}A_{14})^{m_4} = 1$$

$$(A_{34}A_{24}A_{14}A_{12}A_{34}^{-1})^{m_3} = 1$$

$$(A_{23}A_{34}A_{24}A_{45}A_{35}A_{25}A_{12}A_{35}^{-1}A_{45}^{-1}A_{34}^{-1})^{m_2} = 1 (A_{23}A_{34}A_{24}A_{45}A_{35}A_{25})^{m_1} = 1.$$

2. $\Gamma = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1x_2x_3x_4x_5 = x_1^{m_1} = 1$

$1 \leqslant i \leqslant 5, m_i \neq m_j, \text{ for } i \neq j \rangle = x_i^{m_i} = 1$

$Aut^+ \Gamma$ is generated by A_{ij}, $1 \leqslant i < j \leqslant 5$, with defining relations (1.2) and

$$A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{45}A_{35}A_{25}A_{15} = 1$$

$$(A_{45}A_{35}A_{25}A_{15})^{m_5} = 1$$

$$(A_{45}A_{35}A_{25}A_{15}A_{12}A_{23}A_{13}A_{45}^{-1})^{m_4} = 1$$

$$(A_{34}A_{45}A_{35}A_{23}A_{13}A_{45}^{-1})^{m_3} = 1$$

$$(A_{23}A_{34}A_{24}A_{45}A_{35}A_{25}A_{12}A_{35}^{-1}A_{45}^{-1}A_{34}^{-1})^{m_2} = 1 \quad (A_{23}A_{34}A_{24}A_{45}A_{35}A_{25})^{m_1} = 1.$$

3. $\Gamma = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1x_2x_3x_4x_5x_6 = x_i^{m_i} = 1. 1 \leqslant i \leqslant 6, m_i \neq m_j \text{ for } 1 \neq j \rangle$.

$Aut^+ \Gamma$ is generated by A_{ij}, $1 \leqslant i < j \leqslant 6$, with defining relations (1.2) and

$$A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{45}A_{35}A_{25}A_{15}A_{56}A_{46}A_{36}A_{26}A_{16} = 1$$

$$(A_{56}A_{46}A_{36}A_{26}A_{16})^{m_6} = 1$$

$$(A_{56}A_{46}A_{36}A_{26}A_{16}A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{56}^{-1})^{m_5} = 1$$

$$(A_{45}A_{56}A_{46}A_{12}^{-1}A_{13}^{-1}A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{56}^{-1})^{m_4} = 1$$

$$(A_{34}A_{45}A_{35}A_{56}A_{46}A_{36}A_{12}^{-1}A_{23}A_{12}A_{46}^{-1}A_{56}^{-1}A_{45}^{-1})^{m_3} = 1$$

$$(A_{23}A_{34}A_{24}A_{45}A_{35}A_{25}A_{56}A_{46}A_{36}A_{26}A_{12}A_{36}^{-1}A_{46}^{-1}A_{56}^{-1}A_{35}^{-1}A_{45}^{-1}A_{34}^{-1})^{m_2} = 1$$

$$(A_{23}A_{34}A_{24}A_{45}A_{35}A_{25}A_{56}A_{46}A_{36}A_{26})^{m_1} = 1$$

Remarks 2.4.

I. We are unable to find the general formulae for $I(x_i^{m_i})$, since our technique is iterative. However, given a particular r, one can calculate $I(x_i^{m_i})$.

II. If Γ is a triangle group with distinct periods, then

$$\mathrm{Aut}^+\,\Gamma = P_3^* = I(\Gamma).$$

3. PURE MAPPING CLASS GROUPS

The mapping class group can be looked upon as the quotient group of the orientation-preserving automorphisms, $\mathrm{Aut}^+\,\Gamma$, of a Fuchsian group, Γ, by its normal subgroup of inner automorphisms, (Maclachulan and Harvey, 1975). Corresponding to the Fuchsian group of genuz zero with r distinct periods, we can get the pure mapping class group, denoted by PM(0, r). So much has been said in the past about the full mapping class groups, (Birman, 1974), but we cannot find much information about the pure mapping class groups.

In this section, we will give the presentations of PM(0, r), based on the calculations in the examples. The technique is to set the terms within the brackets, that is the terms with periods, equal to one, since they are either $I(x_i)$ or $(I(x_i^{-1})$. Then we reduce these relations to the simplified form.

3.1. PM(0, 3) = 1.

(Trivial form remark 2.4.)

3.2

PM(0, 4) is generated by A_{ij}, $1 \leqslant i < j \leqslant 4$, with defining relations (1.2) and

$$A_{34}A_{23}A_{13} = 1$$

$$A_{34}A_{24}A_{14} = 1$$

$$A_{12}A_{34}^{-1} = 1$$

$$A_{23}A_{34}A_{24} = 1$$

3.3

PM(0, 5) is generated by A_{ij}, $1 \leqslant i < j \leqslant 5$, with defining relations (1.2) and

$$A_{45}A_{34}A_{24}A_{14} = 1$$

$$A_{45}A_{35}A_{25}A_{15} = 1$$

$$A_{12}A_{23}A_{13}A_{45}^{-1} = 1$$

$$A_{34}A_{45}A_{35}A_{12}^{-1} = 1$$

$$A_{23}A_{34}A_{24}A_{45}A_{35}A_{25} = 1$$

3.4

PM(0, 6) is generated by A_{ij}, $1 \leqslant i < j \leqslant 6$, with defining relations (1.2) and

$$A_{56}A_{45}A_{35}A_{25}A_{15} = 1$$

$$A_{56}A_{46}A_{36}A_{26}A_{16} = 1$$

$$A_{12}A_{23}A_{13}A_{34}A_{24}A_{14}A_{56}^{-1} = 1$$

$$A_{23}A_{13}A_{12}A_{46}^{-1}A_{56}^{-1}A_{45}^{-1} = 1$$

$$A_{34}A_{45}A_{35}A_{56}A_{46}A_{36}A_{12}^{-1} = 1$$

$$A_{23}A_{34}A_{25}A_{45}A_{35}A_{25}A_{56}A_{46}A_{36}A_{26} = 1$$

Remarks 3.1.

If Γ is a Fuchsian group with signature (2.4), then $\mathrm{Aut}^+ \Gamma/I(\Gamma)$ is isomorphic to the mapping class group corresponding to the symmetric group $\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i}$.

$$\left\{ I(x_i) : \ell_n + 1 \leqslant i \leqslant \ell_{n+1}, 0 \leqslant n \leqslant k-1 \right\}$$

in terms of the generators of $\eta_1 \upsilon^{-1} (\overset{k}{\underset{i=1}{\pi}} \Sigma_{\alpha_i})$, we can determine the presentation of this mapping class group. This mapping class group lies in between the pure mapping class group and the full mapping class group.

4. SEIFERT FIBRE GROUPS

Let Γ be a Fuchsian group:

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, x_1, x_2, \ldots, x_r | x_i^{m_i}$$

$$= \overset{r}{\underset{i=1}{\pi}} x_i \overset{g}{\underset{j=1}{\pi}} [a_j, b_j] = 1 \rangle$$

Let G be a central extension, by Γ, of Z

$$1 \longrightarrow (z) \rightarrowtail G \overset{\psi}{\longrightarrow} \psi \; \Gamma \to 1 \quad (4.1)$$

such that:

$$G = \langle a_1, b_1, \ldots, a_g, b_g, x_1, x_2, \ldots, x_r, z |$$

$$x_i^{m_i} z^{n_i} = 1,$$

$$\overset{r}{\underset{i=1}{\pi}} x_i \overset{g}{\underset{j=1}{\pi}} [a_j, b_j] = z^n,$$

$$z \leftrightarrow x_i, a_j, b_j \rangle \qquad (4.2)$$

where $\longleftrightarrow$ denotes commutativity.

In Orlik's notation, (1972), we restrict ourselves to the case $0_1 : \epsilon_i = 1$ for all i. If for each, i, $1 \leqslant i \leqslant r$, (m_i, n_i) are relatively prime positive integers and $0 < n_i < m_i$, then $G = \pi_1(M)$, where M is a Seifert manifold. We call G a *Seifert fibre group.* We call the *signature* of M as: $\left\{ n; g; (m_1, n_1), (M_2, n_2), \ldots, (m_r, n_r) \right\}$

We call M *small* if it satisfies one of the following:

(i) $g = 0, r < 2$

(ii) $g = 0, r = 3, 1/m_1 + 1/m_2 + 1/m_3 > 1.$

(iii) $[-2; 0; (2,1), (2,1), (2,1), (2,1)]$

(iv) $g = 1, r = 1.$

Otherwise, we call M *large.*

We summarize below a special case of Orlik s theorem, [10], restricted to the case $0_1 : \epsilon_1 = 1$ for all i.

Theorem 4.1.

Let M and M' be large 0_1 – Seifert manifolds. If $\phi : G' = \pi_1(M') \to G = \pi_1(M)$ is an isomorphism with $z' \to z$, then $g' = g$, $r' = r$, $m_i' = m_i$. $n_i' = n_i$, $(\lambda = 0)$ for all i, and $\phi(x_i') = \ell_i x_{\mu_i} \ell_i^{-1}$, $1 \leqslant i \leqslant r$, where

$\begin{pmatrix} 1 & 2 & \ldots & r \\ \mu_1 & \mu_2 & \ldots & \mu_r \end{pmatrix}$ is a permutation, $m_i = m_{\mu_i}$, $\ell_i \in G$.

Corollary 4.1

Let M be a large 0_1 – Seifert manifold with signature $\left\{ n; 0; (M_1, n_1), (m_2, n_2), \ldots, (m_r, n_r) \right\}$. Then an automorphism $A^* : G \to G$ such that $A^*(z) = z$ satisfies:

$$A^*(x_i) = \ell_i x_{\mu_i} \ell_i^{-1}, 1 \leqslant i \leqslant r,$$

where

$\begin{pmatrix} 1 & 2 & \ldots & r \\ \mu_1 & \mu_2 & \ldots & \mu_r \end{pmatrix}$ is a permutation,

$m_i = m_{\mu_i}$ and $\ell_i \in G$.

Proof

Set $M' = M$ in Theorem 4.1. for $g = 0$.

We denote those automorphisms which satisfy Corollary 4.1. by $\mathrm{Aut}^+ G$, which form a subgroup of Aut G. We call the element $A^* \in \mathrm{Aut}^+ G$, a *regular automorphism*

Theorem 4.2

Suppose G and Γ are as (4.1) and (4.2), respectively, for $g = 0$. Then $\mathrm{Aut}^+ G \cong \mathrm{Aut}^+ \Gamma$.

Proof

By Zieschang's theorem (1966) $A \in Aut^{+} \Gamma$ satisfies:

$$A(x_i) = \ell_i x_{\mu_i} \ell_i^{-1}, \quad 1 \leqslant i < r,$$

where

$$\begin{pmatrix} 1 & 2 \dots r \\ \mu_1 & \mu_2 \dots \mu_r \end{pmatrix} \text{ is a permutation, } m_i = m_{\mu_i}, \ell_i \in \Gamma$$

Let $\psi: G \to \Gamma$. Then ψ induces $\psi_*: Aut^{+} G \to Aut^{+} \Gamma$, $\psi_*(A^*) = A$ and $\ker(\psi_*)$ trivial. Hence, $Aut^{+} G \cong Aut^{+} \Gamma$

Corollary 4.2.

$$\begin{aligned} Out^{+} G = Aut^{+} G/I(G) &\cong Aut^{+} \Gamma/I(\Gamma) \\ &\cong \text{Mapping class group of a closed orientable surface, } X_0 \text{, of genus zero such that } X_0 = \pi_1(\Gamma). \end{aligned}$$

Proof:

Observe that $I(G) \cong G/(z) \cong \Gamma \cong I(\Gamma)$ and $\psi_*(I(G)) = I(\Gamma)$. Therefore, $\widehat{\psi}_*: Aut^{+} G \to Aut^{+}\Gamma/I(\Gamma)$ has $\ker \widehat{\psi}_* = I(G)$. Hence the results follow.

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