Maximum Likelihood Performance of Mean Time to Failure for Right-Censored Data

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Abstract

In this paper failure times following Weibull, exponential and log-normal distribution are considered. The parameters of these distributions are estimated by the maximum likelihood method and these values are used to estimate other quantity of interest such as Mean Time to Failure (MTTF), an important function in a reliability analysis. This study is to look at the performance of maximum likelihood estimate (MLE) under various conditions by considering varying sample size and percentage of censored data. The performance is quantified from the study.

Introduction

Manufacturing industry plays an important role in developing a country's economy. Considering this, it is utmost important for a manufacturer to develop new products in a short period of time using modern technologies and continue to improve the quality of the products. Thus reliability improvement in products is indeed required in a quality improvement program. As noted by [1] and [2], the feasibility of product improvement depends on the series of tasks to be carried out in order to evaluate aspects related to product performance.

Within a short period of time, data connected to product's reliability must be collected and analyzed. The result from the analysis should be used in the improvement of the current ones and new product development. The analysis of failure time data is one of the methodologies that can be used to evaluate the performance of the products in its operational lifetime. This analysis consists of modelling the failure lifetime data using some underlying distributions (e.g. Weibull and log-normal). In most statistical literature, reliability is defined as the probability that a system or item will function over some period of time under specified condition [3]. In this sense, the reliability depends directly on time (failure time) but this concept is not easily assimilated in some situations and to overcome this difficulty, the reliability may be reported by for example mean time to failure (MTTF) if the item is not repairable [4].

The focus of this study is to look at the maximum likelihood estimates of MTTF under the exponential, Weibull and log-normal distributions, some of the common used distributions in reliability studies. It is also known that estimates of these parameters are biased when small samples are used and such bias increases if some censored observations are present. Experiments in reliability usually involved few sample units and the presence of censored data is common phenomena.

The Likelihood Function

Let $T_1^0,\ldots,T_n^0$ be the true failure times of a sample of size n, assumed to be independent identically distributed (*i.i.d*) with distributions exponential, Weibull or log-normal. Assuming that these observations are subject to arbitrary right censoring, the period of follow-up for the *i-th* individual is limited to a value C_i. Here $C_1,\ldots,C_n$ are assumed to be *i.i.d* with a uniform distribution. Then, the observed failure time of the *i-th* individual is given by $T_i=\min\left(T_i^0, C_i\right)$. Together with this is the indicator variable, δ_i which is defined as $\delta_i=0$ if $T_i^0 \ge C_i$ (censored) and $\delta_i=1$ if $T_i^0 < C_i$ (observed failure). If $t_1,t_2,\ldots t_n$ are the values of a random sample from a population with parameter θ, the likelihood function of the sample is

$$L(\theta)=\prod_{i=1}^{n} f(t_i;\theta)$$

for values of θ within a given domain where $\prod_{i=1}^{n} f(t_i;\theta)=f(t_1,t_2,\ldots t_n;\theta)$ is the value of the joint probability distribution or the joint probability density of the random variable $T_1,T_2,\ldots,T_n$ at $T_1=t_1,T_2=t_2,\ldots,T_n=t_n$.

Censored observations make contributions to the likelihood which is a function of the reliability function.

An observation which is right-censored at t_c contributes to the $R(t_c;\theta)$. For example, a given data consisting of a set U of uncensored observations and a set S of right-censored observations, the likelihood function is

$$\prod_{i\in U} f(t_i;\theta)\prod_{i\in S} R(t_i;\theta) \qquad (1)$$

The method of maximum likelihood consists of maximizing the likelihood function with respect to θ.

The value of θ that maximizes the likelihood function *is the maximum likelihood estimate of θ.*

Exponential Distribution

The probability density function (pdf) of an exponential distribution has the form

$$f(t,\lambda) = \lambda e^{-\lambda t}, \quad t \geq 0$$

where $\lambda > 0$ is a parameter of the distribution, often called rate parameter. The distribution is supported on the interval $[0, \infty)$.

The reliability function for exponential distribution is given as

$$R(t;\lambda) = e^{-\lambda t}, \quad t \geq 0$$

The mean or expected value and the variance of an exponential distributed random variable T with rate parameter X are given by

$$MTTF = E(T) = \frac{1}{\lambda} \text{ and } V(T) = \frac{1}{\lambda^2} \quad .$$

Standard error for mean of T is

$$SE = \sqrt{\frac{V(T)}{n}} \quad .$$

Weibull Distribution

The pdf of Weibull distribution is

$$f(t;k,\lambda) = \frac{k}{\lambda^k} t^{k-1} \exp\left[-\left(\frac{t}{k}\right)^k\right], \quad t \geq 0$$

where $k > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. When $k = 3.4$, then the Weibull distribution appears similar to the normal distribution. When $k = 1$, it reduces to the exponential distribution.

The reliability function for Weibull distribution is

$$R(t;k,\lambda) = \exp\left[-\left(\frac{t}{\lambda}\right)^k\right], \quad t \geq 0$$

$$MTTF = E(T) = \lambda\Gamma(1 + 1/k)$$

$$V(T) = \lambda^2\Gamma(1 + 2/k) - E(T)^2$$

The standard error is

$$SE = \sqrt{\frac{\lambda^2\Gamma(1+2/k) - E(T)^2}{n}} \quad .$$

Log-Normal Distribution

The log-normal distribution has pdf

$$f(t;\mu,\sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln(t)-\mu)^2}{2\sigma^2}\right], \quad t > 0$$

where μ, the mean and σ, the standard deviation, are log-normal parameters.

The reliability function for log-normal is

$$R(t;\mu,\sigma) = 1 - \phi\left(\frac{\ln(t)-\mu}{\sigma}\right), \quad t \geq 0$$

$$MTTF = E(T) = e^{\mu + \frac{\sigma^2}{2}}$$

$$V(T) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

The standard error for mean of T is

$$SE = \sqrt{\frac{(e^{\sigma^2} - 1)e^{2\mu+\sigma^2}}{n}} \quad .$$

By employing (1) for all the three distributions the estimated mean time to failure, MTTF can be obtained and the standard error calculated. To incorporate the percentage of censoring, following [6] the joint probability density function for T (survival time) and $T^{\circ}C$ (censoring time), both independent of each other is

$$g_{T,C}(t, c) = f_T(t) \cdot f_C(c) \quad .$$

Suppose that the censoring times follow Uniform $(0,b)$, then the censoring parameter b will be chosen such that it results in an overall probability of censoring $P_c = P(C < T)$ $.P_c$

$$P_c = P(C < T) \qquad (2)$$

$$= \int_0^\infty \int_0^t \frac{1}{b} f(t)\, dc\, dt \; .$$

Generated Data

In this section a comparison study was carried out to evaluate the performance of MLE of MTTF for exponential, Weibull and log-normal distributions. For all three distributions the sample size considered were $n = 20, 50$ and 100, and the percentage of censoring were 0, 10 and 25. The parameter values considered were as follows:

Exponential : $\lambda = 0.01$

Weibull : $\lambda = 1000, k = 5$

Log-normal : $\mu = 4, \sigma = 1.5$

Three sets of data were generated from each distribution. For the exponential distribution with $\lambda = 0.01$, the values of $b = 1000(P_c = 0.1)$ and $b = 400(P_c = 0.25)$, where b was obtained by utilizing (2). For Weibull distribution for each n generated with $\lambda = 1000$ and $k = 5$, the values of b are 9181.69 and 3672.676 for $P_c = 0.1$ and $P_c = 0.25$ respectively. For log-normal distribution with $\mu = 4$ $\mu = 4$ and $\sigma = 1.5$ $\sigma = 1.5$, the values for b are 1681.74 $(P_c = 0.1)$ and 672.696 $(P_c = 0.25)$. Each data were replicated 100 times. The results are shown in Table 1.

Table 1 : $\hat{MTTF}$ and $\hat{SE}$ for the Exponential, Weibull and Long-Normal distributions.

Sample size	Percentage of Censored Data	Exponential $\hat{MTTF}$	Exponential $\hat{SE}$	Weibull $\hat{MTTF}$	Weibull $\hat{SE}$	Long-Normal $\hat{MTTF}$	Long-Normal $\hat{SE}$
n =20	0	125.9	28.15	914.5	40.46	149.5	67.32
	10	109.9	25.21	878.6	47.34	149.2	56.43
	25	108.5	28.03	975.0	39.40	117.9	55.21
n =50	0	95.1	13.45	910.7	31.74	159.5	54.41
	10	101.5	15.14	958.0	28.67	133.6	36.30
	25	93.4	14.96	918.6	36.71	157.5	69.07
n =100	0	100.6	10.06	925.2	21.51	174.7	50.01
	10	99.3	10.35	902.9	22.08	157.0	42.05
	25	94.5	10.84	917.3	23.54	126.2	35.69

The true MTTF for exponential, Weibull and log-normal are 100, 918 and 168 respectively.

From Table 1, for the exponential distribution, the $\hat{MTTF}$ decreases with the increase of sample size and percentage of censored data. However the SE did not behave in the same manner. Even though the values of $\hat{SE}$ are seen to be decreasing as sample size increases, with a bigger percentage of censored data the values tend to increase. For $n = 100$ with complete failure time, $\hat{MTTF}$ is closest to the true value with the smallest standard error.

There is no obvious trend in the Weibull distribution. Generally the $\hat{SE}$ decreases as n increases, but tend to increase with an increase of censoring percentage. $\hat{MTTF}$ is closest to the true value with $n = 50$ and $P_c = 25\%$. For the log-normal distribution, $\hat{SE}$ decreases as n increases except for n =50 with $P_c = 25\%$. No $\hat{MTTF}$ is actually close to the true value. Overall the exponential distribution gives the smallest SE.

Conclusion

This study illustrated the performance of three distributions, exponential, Weibull and log-normal of failure time data with censored observations. Maximum likelihood method was used in the estimation and these values were used in the estimation of MTTF, an important component in reliability. This study was at a preliminary stage. An extensive simulation study should be carried out with the computation of Mean Square Error (MSE), bias and construction of confidence interval to give a better insight and results of the performance of MLE. Other statistical tests can also be employed to determine the best model.

References

[1] Nelson, W. 1982. *Applied Life Data Analysis.* John Wiley & Sons, New York.

[2] Nelson, W. 1990. *Accelerated Testing, Statistical Models, Test Plans and Data Analysis.* John Wiley & Sons, New York.

[3] Lewis, E.E. 1996. *Introduction To Reliability Engineering.* 2nd Ed. John Wiley & Sons, New York.

[4] Ho, L.L and Da Silva, A.F 2005. Bias Correction for Mean Time to Failure and P-quantiles in a Weibull Distribution by Bootstrap Procedure. *Communication in Statistic-Simulation and Computation,* **34**, 617-629

[5] Wolstenholme, L.C. 1990. *Reliability Modelling, A Statistical Approach.* Boca Raton : Chapman & Hall / CRC

[6] Noor Akma Ibrahim and Abdul Kudus. 2007. Bootstrap Confidence Interval for The Median Failure Time of Three-Parameter Weibull Distribution. *World Congress on Engineering Proceedings,* 836-839.