On Optimality of Pursuit Time

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Abstract

We study a differential game described by an infinite system of differential equations with integral constraints on controls of players. This system is obtained by a parabolic equation by using decomposition method. We obtained an equation to find the optimal pursuit time and examined the series representing the left hand side of the equation. Moreover necessary and sufficient condition for convergence of the series is obtained.

Introduction

Differential games, first considered in the book of Isaacs [1], now present a wide field of research. A number of books on this subject have been published (for instance, [1-5]).

Differential game problems for systems described by partial differential equations are also of increasing interest (see, e.g.,[6], [9], [10]). Some of the control problems for parabolic and hyperbolic partial differential equations can be reduced to the ones described by infinite systems of ordinary differential equations by using decomposition method [7,8]. In [9] control and differential game problems described by the following infinite system of differential equations:

$$\dot{z}_k = -\lambda_k z_k + w_k, \qquad (1)$$
$$z_k(t_0) = z_k^0, \ \ k = 1,2,...,$$

were investigated, where $z_k, w_k \in R^1$, w_k, $k = 1,2,...,$ are control parameters,

$$\lambda_k, 0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_k \le ... \to \infty,$$

are generalized eigenvalues of the elliptic operator

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x) \frac{\partial z}{\partial x_j} \right].$$

The paper [10] presents a detailed analysis and solution of the optimal pursuit problem described by system (1) with integral constraints on controls of players. In the paper by Satimov and Tukhtasinov [9], under certain conditions differential games of kind described by the same system were considered. So there is a significant relationship between control problems described by partial differential equations and those described by infinite system of differential equations.

This paper is concerned with the pursuit differential game in the Hilbert space l_r^2 with integral constraints on the control functions of players. It is not necessary that the sequence of positive numbers λ_k is increasing as required in the papers [9] and [10].

The Pursuer tries to force the states of the system associated with the differential game toward the origin of the space l_r^2 against any action of the opposite player, the Evader, who exactly tries to avoid this. It is obtained an equation to find the optimal pursuit time and examined the series representing the left hand side of the equation. It is also obtained necessary and sufficient condition for convergence of the series.

Statement of the problem

Let $\lambda_1, \lambda_2, ...$ be a sequence of positive numbers, and r be a fixed number. The sequence $\lambda_1, \lambda_2, ...$ need not satisfy the requirement that $\lambda_1 \le \lambda_2 \le ...$. We introduce into the consideration the space

$$l_r^2 = \{\alpha = (\alpha_1, \alpha_2, ...) : \sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 < \infty\}$$

with the inner product and the norm

$$(\alpha,\beta)_r = \sum_{i=1}^{\infty} \lambda_i^r \alpha_i \beta_i, \alpha, \beta \in l_r^2,$$

$$\| \alpha \| = \left(\sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 \right)^{1/2}$$

Let $L_2(t_0, T; l_r^2)$ be the space of the functions

$$f(t) = (f_1(t), f_2(t), ...), t_0 \le t \le T,$$

with measurable coordinates $f_k(t), t_0 \le t \le T$, satisfying the inequality

$$\sum_{k=1}^{\infty} \lambda_k^r \int_{t_0}^{T} \| f_k(t) \|^2 \, dt < \infty.$$

We consider differential game described by the infinite system of of differential equations

$$\dot{z}_k = -\lambda_k z_k - u_k + v_k, \quad (2)$$
$$z_k(t_0) = z_k^0, k = 1, 2, ...,$$

where

$$z_k, u_k, v_k, t_0, z_k^0 \in R^1,$$
$$z^0 = (z_1^0, z_2^0, ...) \in l_{r+1}^2, z^0 \ne 0,$$

$u = (u_1, u_2, ...)$ is control parameter of the pursuer and $v = (v_1, v_2, ...)$ is that of the evader. Suppose that $u(\cdot), v(\cdot) \in L_2(t_0, T; l_r^2)$, where T is fixed positive number.

Definition 1: A function $u(\cdot) = (u_1(\cdot), u_2(\cdot), ...)$, satisfying the condition

$$\int_{t_0}^{T} \| u(t) \|^2 \, dt \le \rho^2, \ \| u(t) \| = \left[\sum_{k=1}^{\infty} \lambda_k^r u_k^2(t) \right]^{1/2},$$

where $\rho_1, \rho_2, ...$ are given positive numbers, is called the admissible control of the pursuer.

Definition 2: A function $v(\cdot) = (v_1(\cdot), v_2(\cdot), ...)$, satisfying the condition

$$\int_{t_0}^{T} \| v(t) \|^2 \, dt \le \sigma^2, \ \| v(t) \| = \left[\sum_{k=1}^{\infty} \lambda_k^r v_k^2(t) \right]^{1/2},$$

where σ is given number, is called the admissible control of evader.

Definition 3: Suppose that

$$w(\cdot) = (w_1(\cdot), w_2(\cdot), ...) \in L_2(t_0, T; l_r^2),$$
$$(z_{10}, z_{20}, ...) \in l_{r+1}^2.$$

A function

$$z(t) = (z_1(t), z_2(t), ...), t_0 \le t \le T,$$

is called the solution of the system of equations

$$\dot{z}_k = -\lambda_k z_k + w_k, z_k(t_0) = z_k^0, k = 1, 2, ...,$$

if

1) each coordinate $z_k(t)$ is absolutely continuous function and satisfies the equation (3) almost everywhere on $[t_0; T]$,

2) $z(\cdot) \in C(t_0, T; l_{r+1}^2)$, where $C(t_0, T; l_{r+1}^2)$ is the space of continuous functions

$$z(t) = (z_1(t), z_2(t), ...), \ t_0 \le t \le T,$$

with the values in l_{r+1}^2.

It was shown [11] that for any given positive number T the system (1) has a unique solution

$$z(t) = (z_1(t), z_2(t), ...), t_0 \le t \le T,$$

of course

$$z_k(t) = z_{k0} e^{-\lambda_k (t - t_0)} + \int_{t_0}^{t} e^{-\lambda_k (t-\tau)} w_k(\tau) d\tau,$$

$t \in [t_0; T], \ k = 1, 2, ...$

Definition 4: A function of the form

$$u(t, v) = w_0 + v, 0 \le t \le T,$$

is called strategy of the pursuer, where

$$w_0(\cdot) = (w_{10}, w_{20}, ...) \in L_2(0, T; l_r^2)$$

is arbitrary function satisfying

$$\left[\sum_{k=1}^{\infty} \lambda_k^r \int_0^T w_{k0}^2(t) dt \right]^{1/2} \le \rho - \sigma,$$

Definition 5: We say that pursuit starting from the initial position $z_0, z_0 \in l^2_{r+1}$, can be completed for the time T if there exists a strategy $u(t,v)$ of the pursuer such that $z(\tau)=0$, at some τ, $0 \le \tau \le T$, for any evader's control $v(t)$, $0 \le t \le T$.

Definition 6: A function

$$V(u_1, u_2, ...), V : l^2_r \times l^2_r \times ... \to l^2_r,$$

is referred to as the strategy of the evader if

1) for any admissible control of the pursuer $u = u(t)$, $t_0 \le t \le T$, the system (2) has a unique solution at

$$v = V(u_1(t), u_2(t), ...),\ t_0 \le t \le T;$$

2) the inequality

$$\int_{t_0}^{T} \| V(u_1(t), u_2(t), ...) \|^2 \, dt \le \sigma^2$$

holds.

Definition 7. A number ϑ is called the optimal pursuit time for the initial position $z_0, z_0 \in l^2_{r+1}$, in the game (2) if

i) pursuit can be completed for the time ϑ,
ii) there exists a strategy of the evader such that $z(t) \ne 0$, $t \in [t_0, \vartheta)$.

Results and Discussion

It is true the following theorem.

Theorem *Let* $\rho > \sigma$ *and* $z_0 \in l^2_{r+1}$. *If* $z_0 \in l^2_r$, *then equation*

$$\sum_{k=1}^{\infty} \lambda_k^r R_k(t) z_{k0}^2 = (\rho - \sigma)^2,$$
$$R_k(t) = 2\lambda_k / \left(e^{2\lambda_k t} - 1\right)$$

has a unique solution $t = \vartheta$ *and it is optimal pursuit time.*

For the convergence of the series in this theorem we can prove the following lemma.

Lemma *Let* $z_0 \in l^2_{r+1}$. *The series*

$$I = \sum_{k=1}^{\infty} \lambda_k^r R_k(t) z_{k0}^2$$

converges for any $t > 0$ *if and only if* $z_0 \in l^2_r$.

The theorem gives a solution to the important special case of the Theory of Differential Games, the optimal pursuit problem. First, we have found formula to the optimal pursuit problem. Secondly, that we have constructed optimal strategies for players. Finally, we have obtained necessary and sufficient condition for convergence the series presenting the left-hand side of the equation for the optimal pursuit time.

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