The Uniform Convergence of Spectral Expansions of the Laplace Operator on Closed Domain

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Abstract

In this paper we investigate the uniform convergence of spectral expansions of the biharmonic operator on closed domain $\bar{\Omega} \subset \mathbb{R}^2$. New estimates for the spectral function of the biharmonic operator on closed domain are obtained.

Introduction

In early twentieth century, connection between the theory of multiple Fourier series and the theory of partial differential equations was found. The partial sums of multiple Fourier series coincide with the spectral decomposition associated with the Laplace operator on the torus. Such spectral approach to the study of multiple Fourier series gave impetus to the development of spectral theory of differential operators.

Justification of applicability of separation of variables for solving equations of mathema-tical physics has become an important part in mathematical sciences. The formation and development of the theory of spectral expansions related to differential operators became possible, thanks to the work of scholars such as Titchmarsh, Bochner, Stein, Il'in , Hormander and Alimov [1, 3-5, 8, 9]. The most complete study of problems of summability and convergence of spectral expansions related to the differential operators can be found in the works of V.A. Il'in and his disciples (see e.g. [3-5]). In their work, they conducted in depth studies of spectral expansions related to self-adjoint and non self-adjoint differential operators and to operators with singular coefficients. They established new methods to investigate the spectral expansions based on estimates of a fundamental system of functions, spectral functions and their Riesz means, all of which are reliable tools of investigation.

Problems of convergence and summability of spectral decomposition related to elliptic differential operators are important from the point of view of development of the mathematical basis of modern mathematical physics. In this note, we discuss the case of biharmonic operator whose applications in continuum mechanics are known.

Main results

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$. We denote the eigenfunctions and eigenvalues of the biharmonic operator Δ^2 by $u_n(x,y)$ and λ_n respectively where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator:

$$\begin{aligned} \Delta^2 u_k(x) - \lambda_k u_k(x) &= 0, \\ u_k\big|_{\partial\Omega} = 0 &= \Delta u_k\big|_{\partial\Omega}. \end{aligned} \quad (1)$$

We will investigate the uniform convergence of the biharmonic operator on a closed domain $\bar{\Omega} \subset \mathbb{R}^2$.

We proceed to the formulation of the fundamental results of this paper.

Theorem 1: *For $u_n(x,y)$ and λ_n ,respectively the eigenfunctions and eigenvalues of the biharmonic operator Δ^2 corresponding to the boundary conditions $u_k\big|_{\partial\Omega} = 0 = \Delta u_k\big|_{\partial\Omega}$, we have*

$$\sum_{\mu < \sqrt[4]{\lambda_n} \le \mu+1} u_n^2(x,y) \le C \cdot (\mu+1)\ln^2(\mu+1), \quad (2)$$

for all $(x,y) \in \bar{\Omega} = \Omega \cup \partial\Omega$.

Firstly in [3], Il'in has obtained conditions for uniform and absolute convergence of decomposition in terms of eigenfunctions on the closed domain for any of the three boundary problems of the Laplace operator. In particular it is proved that a Fourier series of a function from a class $W_p^{(\frac{1}{2}N+1)}(\Omega)$ (we denote Sobolev space by $W_p^l(\Omega)$), satisfying the repeated Laplacian equation and boundary conditions, uniformly converge in the closed domain $\bar{\Omega}$.

In [10], Eskin has shown that if $f \in \overset{\circ}{W}{}_p^{(l)}(\Omega)$, $\lambda = (N-\frac{1}{2})/2+\varepsilon$ where $\varepsilon > 0$, then the decomposition of f in terms of eigenfunctions of the elliptic operator of order $2m$ with the boundary conditions of Lopatinsky, uniformly converges in the closed domain $\bar{\Omega}$. In [11], Moiseyev investigated problems of uniform convergence in the closed domain $\bar{\Omega}$ of decomposition of the elliptic operator of the second order. In [11], it is proved that if the expanding function f belongs to the class $W_p^{(N-1)/2}(\Omega)$ with $p > 2N/(N-1)$ and it has compact support in Ω such that the series

$$\sum_{k=1}^{\infty} \lambda_k^{(N-1)/2} \left(\ln \lambda_k\right)^{2+\varepsilon} f_k^2 < \infty$$

where f_k are the Fourier coefficients of f, then the spectral expansions of the Laplace operator related to the first boundary problem uniformly converges on the closed domain $\bar{\Omega}$.

The estimate (2) allows us to prove uniform convergence of spectral expansions of continuous function.

Theorem 2: *Let* $f(x,y)$ *be arbitrary continuous function on* Ω. *Then the Riesz means of spectral expansions*

$$E_\lambda^s f(x,y) = \sum_{\lambda_n < \lambda} \left[1 - \frac{\lambda_n}{\lambda}\right]^s f_n u_n(x,y)$$

of the order $s > \frac{1}{2}$, *converges uniformly to* $f(x,y)$ *on* $\bar{\Omega}$ *where* $u_n(x,y)$ *and* λ_n *are respectively the eigenfunctions and eigenvalues of the biharmonic operator* Δ^2 *satisfying boundary conditions* $u\big|_{\partial\Omega} = 0 = \Delta u\big|_{\partial\Omega}$.

Il'in in [4] has established the sharp conditions for uniform convergence of the spectral expansions. He found conditions on the order of differentiability a, the degree of summability p, and the dimension of space N:

$$a \geq \frac{N-1}{2}, \quad a \cdot p > N, \quad p \geq 1, \qquad (3)$$

where the spectral expansion of the functions from W_p^a with finite support converges uniformly on any compact $K \subset \Omega$. For Riesz means, instead of conditions (3), there should be conditions:

$$a \geq \frac{N-1}{2} - s, \quad a \cdot p > N, \quad p \geq 1. \quad (4)$$

It follows from the theorem of V.A. Il'in (see [4]) that the first of inequalities in (4) is sharp. The inequality $a \cdot p \leq N$ admits an existence of unbounded functions from the class $W_p^a(\Omega)$, the Riesz means of spectral decomposition of which obviously does not converge uniformly (see [5]). Sh.A. Alimov in [1] has established conditions (4) providing uniform convergence of Riesz means of spectral decomposition of functions from S.M. Nikolsky's classes $H_p^a(\Omega)$ for any self-adjoint extension of the elliptic operator. For the general elliptic operator of an order $2m$, it is proved that it is not dependent on the number m and the same conditions (4) provide uniform convergence on any compact $K \subset \Omega$, of Riesz means of the spectral expansions of functions from $H_p^a(\Omega)$.

Proof of Theorem 2

The proof of Theorem 2 is based on the estimate in Theorem 1.

For any $h > 0$ we define a set $\Omega_h \subset \Omega$ where $\Omega_h = \{M(x,y) \in \Omega : \mathrm{dist}(M, \partial\Omega) > h\}$. Let $M_0(x_0, y_0) \in \Omega_h$ and $M(x,y) \in \bar{\Omega}$. Let us consider the radial function $V_\lambda(r) = V_\lambda\left(dist(M_0, M)\right)$:

$$V(r) = \begin{cases} \Gamma(s+1)\,2^s \lambda^{\frac{1-s}{2}} \times \dfrac{J_{s+1}\left(r\sqrt{\lambda}\right)}{2\pi r^{s+1}}, & r \le R \\ 0 & R > 0 \end{cases}$$

where R is less than $h/4$ and $J_\nu(t)$ is the Bessel function of order ν.

For eigenfunctions $u_k(x,y)$ of the biharmonic operator, we have (see [12]):

$$\iint_{r\le R} u_k(x + r\cos\theta, y + r\sin\theta)\, r\, dr\, d\theta = 2\pi R J_1\left(R\nu_k\right)\nu_k^{-2} u_k(x,y) + O\left(e^{-(r_0 - R)\nu_k}\right) \tag{3}$$

where $\nu_k = \sqrt[4]{\lambda_k}$. Note that

$$\int_0^\infty J_{a+s}(\lambda t)\, J_{a-1}(\nu_k t)\, t^{-s}\, dt = \begin{cases} \dfrac{\left[1 - \dfrac{\nu_k^2}{\lambda^2}\right]^s \lambda^{2s} \nu_k^{a-1}}{2^s\,\Gamma(s+1)\,\lambda^{a+s}}, & \nu_k \le \lambda\,. \\ 0 & , \quad \nu_k > \lambda \end{cases} \tag{4}$$

Using the formula (3), we obtain for Fourier coefficients of the function $V_\lambda(r)$ in the next representation

$$V_\lambda(r) = 2^s\,\Gamma(s+1)\,\lambda^{\frac{1-s}{2}} u_n(x,y) \times \int_0^R J_{1+s}(\lambda r)\, J_0(\nu_n r)\, r^{-s}\, dr. \tag{5}$$

Separating the last integral into two parts i.e. $\int_0^\infty - \int_R^\infty$, and using equality (3), we obtain

$$\hat{V}_n(r) = \delta_n^\lambda u_n(x,y)\left[1 - \frac{\nu_n^2}{\lambda^2}\right]^s - 2^s\,\Gamma(s+1)\,\nu_n^{-1}\lambda^{\frac{1}{2}-s} u_n(x,y)\, I(\lambda,\nu_n), \tag{6}$$

where

$$I(\lambda,\nu_n) = (\lambda\cdot\nu_n)^{1/4}\int_R^\infty J_{1+s}(r\lambda)\, J_0(r\nu_n)\, r^{-s}\, dr,$$

and

$$\delta_n^\lambda = \begin{cases} 1, & \nu_n < \lambda \\ 0, & \nu_n \ge \lambda \end{cases}.$$

Multiplying both sides of (6) by $u_n(x,y)$ and we have in a sense of L_2 that

$$V_\lambda(r) = \Theta^s(M_0, M, \lambda) - \left(2^s\,\Gamma(s+1)\,\lambda^{\frac{1}{2}-s} \times \sum_{n=1}^\infty u_n(x,y)\cdot u_n(x_0,y_0)\,\nu_n^{-1}\, I(\lambda,\nu_n)\right),$$

where the function

$$\Theta(M_0, M, \lambda) = \sum_{\lambda_n < \lambda}\left[1 - \frac{\lambda_n}{\lambda}\right]^s u_n(x,y)\cdot u_n(x_0,y_0)$$

is the Riesz means of order s of spectral function of the biharmonic operator. We denote the right side of (2) by $V_\lambda(r)$, where $x \in \Omega_h$, $y \in \bar{\Omega}$.

Using the estimation (2) we obtain

Lemma 1: *For any positive ε, we have*

$$\sum_{\sqrt[4]{\lambda_n} < \lambda} u_n^2(x,y)\,\lambda_n^{\varepsilon - 1} = O\left(\lambda^\varepsilon \ln^2\lambda\right), \quad \forall\,(x,y)\in\bar{\Omega}\,;$$

$$\sum_{\sqrt[4]{\lambda_n} > \lambda} u_n^2(x,y)\,\lambda_n^{-\varepsilon - 1} = O\left(\lambda^{-\varepsilon} \ln^2\lambda\right), \quad \forall\,(x,y)\in\bar{\Omega}\,.$$

Lemma 2: *For all $\lambda > 0$, $\lambda_n > 0$, we have*

$$\iint_\Omega f(x,y)\, V_\lambda(r)\, dx\, dy$$

is a continuous function since $(x,y)\in\bar{\Omega}$.

Let the set Ω_h contains the support of $f(x,y) \in L_2(\Omega)$. By definition of Riesz means, we have

$$
\begin{aligned}
& E_\lambda^s f(x_0,y_0) \\
&= \iint_\Omega f(x,y)\, V_\lambda(r)\, dx\, dy \\
&\quad + \Big(2^s \Gamma(s+1) \lambda^{\frac{1}{4}-\frac{s}{2}} \\
&\quad \times \sum_{n=1}^{\infty} f_n \lambda_n^{\frac{1-N}{4}} u_n(x,y) I(\lambda,\lambda_n)\Big) .
\end{aligned} \tag{7}
$$

Let $B(M_0;R)$ be a ball with the radius R about the point $M_0(x_0,y_0) \in \overline{\Omega}$. Using continuity of function $f(x,y)$,

$$
\begin{aligned}
& \iint_{\Omega_h} f(x,y)\, V_\lambda(r)\, dx\, dy \\
&\quad = 2^s \Gamma(s+1) (2\pi)^{-\frac{1}{2}} \lambda^{\frac{1}{4}-\frac{s}{2}} \\
&\quad \times \int_{\Omega_h \cap B(M_0;R)} J_{1+s}\left(\sqrt{\lambda}\, r\right) \cdot r^{-(1+s)}\, dr
\end{aligned}
$$

and the fact it equals to the first part in (7), we have

$$
\begin{aligned}
& E_\lambda^s f(x_0,y_0) \\
&= \iint_{\Omega_h} f(x,y)\, V_\lambda(|x-y|)\, dx\, dy \\
&\quad + \Big(2^s \Gamma(s+1)\, \lambda^{\frac{1}{4}-\frac{s}{2}} \times \\
&\quad \sum_{n=1}^{\infty} f_n \lambda_n^{-\frac{1}{4}} u_n(x,y) I(\lambda,\nu_n)\Big) .
\end{aligned} \tag{8}
$$

Lemma 3: *For all* $(x,y) \in \overline{\Omega}$, *the inequality*

$$
\sum_{n=1}^{\infty} u_n^2(x,y)\, \lambda_n^2 \left[I(\lambda,\nu_n)\right]^2 \le C \ln^2 \lambda
$$

holds.

Proof: From estimates in Lemma 1 and Lemma 2, it follows that for all $(x,y) \in \overline{\Omega}$,

$$
\sum_{\nu_n \le 1} \nu_n^{-2} \left[I(\lambda,\nu_n)\right]^2 u_n^2(x,y) = O\left(\frac{1}{\lambda}\right).
$$

Similarly, we have for all $(x,y) \in \overline{\Omega}$,

$$
\sum_{1 \le \nu_n \le \frac{\sqrt{\lambda}}{2}} \nu_n^{-2} \left[I(\lambda,\nu_n)\right]^2 u_n^2(x,y) = O\left(\frac{\ln^2 \lambda}{\sqrt{\lambda}}\right) ;
$$

$$
\sum_{\frac{\sqrt{3\lambda}}{2} \le \nu_n} \nu_n^{-2} \left[I(\lambda,\nu_n)\right]^2 u_n^2(x,y) = O\left(\frac{\ln^2 \lambda}{\sqrt{\lambda}}\right) .
$$

For estimating summation corresponding to n for which $\sqrt{\lambda}/2 \le \nu_n \le \sqrt{3\lambda}/2$, we use the estimations in Lemma 1 and Lemma 2. We denote by k the minimum integer for which $2^k \ge \sqrt{\lambda}\, B/2$. Then we obtain for all $(x,y) \in \overline{\Omega}$,

$$
\begin{aligned}
& \sum_{|\nu_n - \lambda| \le \frac{\sqrt{\lambda}}{2}} \nu_n^{-2} \left[I(\lambda,\nu_n)\right]^2 u_n^2(x,y) \\
&\le \sum_{m=1}^{k} \sum_{2^{m-1} \le |\nu_n - \lambda| \le 2^m} \nu_n^{-2} u_n^2(x,y) 4^{1-m} \\
&\le c \ln^2 \lambda ,
\end{aligned}
$$

which proves Lemma 3.

Lemma 4: *If* $s > 1/2$, *then for all continuous functions* $f(x,y)$ *with finite support in* Ω, *we have*

$$
\left| E_\lambda^s f(x,y) \right| \le c \| f \|_\infty, \quad \forall (x,y) \in \overline{\Omega} .
$$

Proof: Using the well-known estimates of Bessel function

$$
|J_\nu(t)| \le \begin{cases} t^{-1/2}, & (t \ge 1) \\ t^{\nu}, & (t \le 1) \end{cases},
$$

we obtain

$$
\begin{aligned}
& \left| \iint_{\Omega_h} f(x,y)\, V_\lambda(r)\, dx\, dy \right| \\
&\le c_1 \| f \|_\infty \left[\int_0^{1/\sqrt{\lambda}} r \cdot |V_\lambda(r)|\, dr \right. \\
&\qquad \left. + \int_{1/\sqrt{\lambda}}^{R} r \cdot |V_\lambda(r)|\, dr \right] .
\end{aligned}
$$

Obviously from estimations of Bessel's functions, it follows that the functions in square brackets are bounded. Now we will estimate the second summation in the right part of equality (7). For this purpose, we apply Holder's inequality and Parseval's formula to the sum

$$
\sum_{n=1}^{\infty} f_n \cdot u_n(x,y) \cdot I(\lambda,\lambda_n)\, \lambda_n^{-\frac{1}{4}} .
$$

Then from Lemma 3, it follows that

$$
\left| \sum_{n=1}^{\infty} f_n\, u_n(x,y) I(\lambda,\lambda_n)\, \lambda_n^{-\frac{1}{4}} \right| \le c \ln \lambda \| f \|_\infty ,
$$

and hence Lemma 4 is proven.

For any function, the Riesz means $E_\lambda^s f(x,y)$ of positive order $s \geq 0$ uniformly converges in the closed domain $\bar{\Omega}$. In view of the density of $C_0^\infty(\Omega)$ in the space $C(\bar{\Omega})$, any function for which requirements of the conditions of Theorem 2 can be approached with the functions from $C_0^\infty(\Omega)$ having $\operatorname{supp}(f) \subset \Omega_h$ where the positive constant h depends only on distance between the function's support and the boundary of domain, then the statement of Theorem 2 follows directly from Lemma 4. Hence Theorem 2 is proved.

Unsolved problems

We would like to pose this open problem for interested readers. Let $\vec{\tau}(x,y)$ be the normal vector to $\partial\Omega$ at point (x,y). Derivative of function $\varphi(x,y)$ on direction $\vec{\tau}(x,y)$ is

$$\frac{\partial\varphi}{\partial\tau} = \frac{\partial\varphi}{\partial x}\cos\left(\widehat{\tau,x}\right) + \frac{\partial\varphi}{\partial y}\sin\left(\widehat{\tau,y}\right).$$

Problem: Let $\varphi_n(x,y)$ and λ_n respectively be eigenfunction and eigenvalues of the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, with the boundary conditions $\partial\varphi/\partial\tau\,|_{\partial\Omega} = 0$. Is the estimate

$$\sum_{\mu<\sqrt{\lambda_n}\leq\mu+1} \varphi_n^2(x,y) \leq C\cdot(\mu+1)\ln^2(\mu+1),$$

true for the eigenfunction of the Laplace operator, satisfying condition $\partial\varphi/\partial\tau\,|_{\partial\Omega} = 0$ for all $(x,y) \in \bar{\Omega} = \Omega \cup \partial\Omega$.

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