Singular Integral Equation for Curved Cracks

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Abstract

The complex potential method is used to formulate singular integral equation for the curved crack problem in plane elasticity. This equation describes the relation of the displacement jump versus the resultant force applied along the crack face.

Introduction

The study of the crack analysis is one of the most important topics in fracture mechanics. Cotterel and Rice [5] used the perturbation procedure to obtain approximate solution for the curved crack correct to the first order in the deviation of the crack surface from a straight line. Panasyuk et. al. [4] developed systems of singular integral equations of plane problems in the theories of elasticity, heat conduction and thermoelasticity for a solid containing a set of straight thermally isolated arbitrary oriented cracks. This includes problem of periodic and doubly periodic cracks. Numerical solution of the obtained singular integral equation was also described.

In this short paper, we will describe the formulation of the singular integral equation for the curved crack problem in plane elasticity based on the complex potential method. The numerical solution for the obtained singular integral equation can be found in many references using various techniques (see for example, [2,3,4,5]).

The singular integral equations

The stresses ($\sigma_x, \sigma_y, \sigma_{xy}$), the resultant forces (X, Y) and the displacements (u, v) are related to complex potential functions $\varphi(z)$ and $\psi(z)$ by [1]

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\Phi(z) \tag{1}$$

$$\sigma_x - i\sigma_y = 2\mathrm{Re}\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)} \tag{2}$$

$$f = -Y + iX = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} \tag{3}$$

$$2G(u+iv) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{4}$$

where G is shear modulus for elasticity, for the plane stress problem $\kappa = (3-v)/(1+v)$ and for the plane strain problem $\kappa = 3-4v$, v is Poisson's ratio. Also note that, we use the notation $\Phi(z) = \varphi'(z)$ and $\Psi(z) = \psi'(z)$ and $\overline{Z}$ is a complex conjugation of Z.

If two point dislocations with intensity $H\left(-H\right)$ are placed at the point $z = t(z = t + dt)$, we obtain the following complex potentials in plane elasticity [2]

$$\left.\begin{aligned} \varphi'(z) &= -H\frac{dt}{t-z} \\ \psi'(z) &= -\overline{H}\frac{dt}{t-z} - H\frac{d\bar{t}}{t-z} + H\frac{\bar{t}dt}{(t-z)^2} \end{aligned}\right\} \tag{5}$$

Substituting $H(\overline{H})$ by $-g(t)/2\pi(-\overline{g(t)}/2\pi)$ in (5) and integrating along the curve L, yields

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi}\int_L \frac{g(t)dt}{t-z} \\
\psi(z) &= \frac{1}{2\pi}\int_L \frac{g(t)d\bar{t}}{t-z} \\
&\quad + \frac{1}{2\pi}\int_L \frac{\overline{g(t)}dt}{t-z} - \frac{1}{2\pi}\int_L \frac{\bar{t}g(t)dt}{(t-z)^2}
\end{aligned}\right\} \qquad (6)$$

where

$$\begin{aligned}
&2G[(u(t)+iv(t))^+ - (u(t)+iv(t))^-] \\
&\quad = 2G(u(t)+iv(t)) = i(\kappa+1)g(t), t \in L
\end{aligned} \qquad (7)$$

which is obtained by substituting (6) into (4) and making use of Plemelj formula [1]. In Equation. (7), $(u(t)+iv(t))^+ - (u(t)+iv(t))^-$ denotes the jump value of the displacement (or the crack opening displacement, COD) for the crack and $(u(t)+iv(t))^+ \; ((u(t)+iv(t))^-)$ is the displacements at point t of the upper (lower) face of the crack.

It is clear that the single-valuedness condition of displacements gives

$$\int_L g(t)dt = 0 \qquad (8)$$

From the first equation of (6), we have $\varphi'(z) = \frac{1}{2\pi}\int_L \frac{g(t)dt}{(t-z)^2}$

and $\overline{\varphi'(z)} = \frac{1}{2\pi}\int_L \frac{\overline{g(t)}d\bar{t}}{(\bar{t}-\bar{z})^2}$. Making use of these terms and substitute (6) into (3), one has

$$\begin{aligned}
f = -Y + iX &= \frac{1}{2\pi}\int_L \frac{g(t)dt}{t-z} + \frac{1}{2\pi}\int_L \frac{z\overline{g(t)}d\bar{t}}{(\bar{t}-\bar{z})^2} \\
&+ \frac{1}{2\pi}\int_L \frac{\overline{g(t)}dt}{\bar{t}-\bar{z}} + \frac{1}{2\pi}\int_L \frac{g(t)d\bar{t}}{\bar{t}-\bar{z}} - \frac{1}{2\pi}\int_L \frac{t\overline{g(t)}d\bar{t}}{(\bar{t}-\bar{z})^2}.
\end{aligned}$$

After simplification, this gives rise to

$$\begin{aligned}
f = -Y + iX &= \frac{1}{\pi}\int_L \frac{g(t)dt}{t-z} + \\
&\frac{1}{2\pi}\int_L g(t)\left(-\frac{1}{t-z} - \frac{1}{\bar{t}-\bar{z}}\frac{d\bar{t}}{dt}\right)dt + \\
&+ \frac{1}{2\pi}\int_L \overline{g(t)}\left(\frac{1}{\bar{t}-\bar{z}} - \frac{t-z}{(\bar{t}-\bar{z})^2}\frac{d\bar{t}}{dt}\right)dt
\end{aligned}$$

Letting $z \to t_0^+, z \to t_0^-$ and $z = t_0$, yield

$$\begin{aligned}
f^+(t_0) = f^-(t_0) &= -Y(t_0) + iX(t_0) \\
&= \frac{1}{\pi}\int_L \frac{g(t)dt}{t-t_0} \\
&+ \frac{1}{2\pi}\int_L g(t)\left(-\frac{1}{t-t_0} - \frac{1}{\bar{t}-\bar{t}_0}\frac{d\bar{t}}{dt}\right)dt \\
&+ \frac{1}{2\pi}\int_L \overline{g(t)}\left(\frac{1}{\bar{t}-\bar{t}_0} - \frac{t-t_0}{(\bar{t}-\bar{t}_0)^2}\frac{d\bar{t}}{dt}\right)dt, \\
&t_0 \in L.
\end{aligned} \qquad (9)$$

Equation (9) is the singular integral equation for the curved crack problem in an infinite plain. It gives the relation with the displacement jump with the resultant force applied along the crack face.

Equation (9) can be solved numerically using various approaches (for example, see [2,3]). Here we have no intention to give a full solution of this equation.

Conclusion

We have shown that the singular integral equation for the curved crack problem can be formulated by the use of complex potential method. Even though, full numerical solutions are not provided here, the obtained singular integral equation is solvable.

References

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