Note on the Products of Distributions

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Abstract

In this study we consider $\mathcal{D}$ the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ the space of distributions defined on $\mathcal{D}$. Now let $f(x,r)$ be distribution in $\mathcal{D}'$ and let $f(x,r)_n = f(x,r)*\delta_n(x,r)$ where δ_n is a certain sequence which converges to the Dirac-delta function. Then the products $f(x,r)\cdot[f(x,s)]$, $[f(x,r)]\cdot[f(x,s)]$ are defined as the limit of the sequences $\{f(x,r)f(x,s)_n\}$, $\{f(x,r)_n f(x,s)_n\}$ provided that the limits h_1, h_2 exist in the sense of

$$\lim_{n\to\infty}\langle f(x,r)f(x,s)_n,\varphi\rangle=\langle h_1,\varphi\rangle,$$

$$\lim_{n\to\infty}\langle f(x,r)_n f(x,s)_n,\varphi\rangle=\langle h_2,\varphi\rangle$$

respectively for all φ in $\mathcal{D}$. In general, two products do not necessarily be equal. In this work, it was proved that two products are equal if they satisfy a property which we call semi-group condition. It was also proved that if products satisfy the semigroup condition then hold the associativity.

Introduction

Distributions were first introduced into science as a result of Dirac's research in quantum mechanics where δ function is systematically used, however the distribution theory appears to have been first formulated in 1936 by S.L., Sobolev [17], where Sobolev defined a distribution of order m on $\mathbb{R}^n$ as a continuous linear form on the space D^m ($\mathbb{R}^n$) of all C^m functions on $\mathbb{R}^n$ with compact support, and gave the main operations such as derivation, multiplication by C^∞ functions as well as regularization, and used them for a study of partial differential equations, later the theory was developed in a symmetric and a precise mathematical sense by L., Schwartz [16], see Zemanian [20].

Thus the theory of distributions is considered an essential progress in the theory of partial differential equations as well as in mathematical physics.

But, in some important cases the theory of distributions fails, such as in the calculation of δ^2 the square of the dirac delta function. Schwartz [16] has proved, in the theory of distribution, there is no reasonable way of introducing the square δ^2 and it is interesting that the symbol δ^2 often appears in quantum mechanics. Similarly there are also some objects such as $H(x)\delta(x)$, $x^{-1}\,\delta(x)$ and $\delta'\delta$ are of special interest in physics and widely used in quantum theory, see [19]. For example, in physics, product of distributions such as $H\delta$ or δ^2 can be interpreted in several different ways; see Colombeau [2]. Thus in the literature, various definitions have been proposed just for δ^2, see Oberguggenberger [15].

There are still many problems in defining the product of singular distributions.

Delta Sequences and Convolution

Definition 1 *A sequence* $\delta_n : \mathbb{R}\to\mathbb{R}$ *is a delta sequence of ordinary functions which converges to the singular distribution* $\delta(x)$ *and satisfy the following conditions* :

(i) $\delta_n(x) \geq 0$ for all $x \in \mathrm{R}$,

(ii) δ_n is a continuous and integrable over R with $\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$

(iii) Given any $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_{-\infty}^{-\varepsilon} \delta_n dx + \int_{\varepsilon}^{\infty} \delta_n(x)dx = 0$$

Example 1 Let $\int_a^b \delta(x) = \dfrac{n}{\pi(n^2t^2+1)}$

then $\int_a^b \delta_n = \dfrac{1}{\pi}[\arctan(nb) - \arctan(an)]$. Then it follows that δ_n is a delta sequence.

Example 2 Let φ be a continuous, nonnegative, $\varphi(x) = 0$ for all $|x| \geq 1$ and $\int_{-1}^{1} \phi = 1$ then set $\delta_n(x) = n\phi(nx)$. Then δ_n is a delta sequence.
As in the above two examples are many ways to construct a delta sequence. In this work we let ρ be a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

The function δ_n is then defined by $\delta_n(x) = n\rho(nx)$ for n = 1,2,.... It follows that $\{\delta_n\}$ is a regular sequence on infinitely differentiable functions converging to the Dirac delta-function δ. If now f is an arbitrary distribution in $\mathcal{D}'$; the function f_n is defined by

$$f(x,r)_n = (f * \delta_n(t) = \langle f(t,r), \delta_n(x-t) \rangle$$

where n = 1, 2,... and r is a fixed parameter. Thus it also follows that $\{f(x,r)_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x,r)$

Product of Distribution

In distribution theory, it is difficult to give exact meaning to product of arbitrary distribution. For example, in the literature there are several definitions just for

$$\delta^2(x) = 0, cPfx^{-2}, c\delta(x), c\delta(x) + \frac{1}{2\pi i}, \delta'(x)$$

to $c\delta(x) + c_1\delta'(x)$ with arbitrary constants c, c_1. Thus, several various definitions on multiplication of generalized functions show that the product of two distributions might have nonstandard value. Known results, generally, depend on the nonstandard representations of distributions chosen.

In the following we give some of these products of distributions. The basis for all later definitions is product of a distribution $f \in \mathcal{D}'(\mathrm{R})$ and an infinitely differentiable function $\varphi \in C^{\infty}(\mathrm{R})$, introduced by L. Schwartz [16] commutatively as the element $f\varphi = \varphi f \in \mathcal{D}'(\mathrm{R})$.

Fourier Transform Method

Given two distributions $f, g \in \mathcal{D}'(\mathrm{R})$ assume that their Fourier transforms $F(f)$, $F(g)$ exists. Then "Fourier Product" is defined as

$$f.g = \mathcal{F}^{-1}[\mathcal{F}(f) * \mathcal{F}(g)]$$

see Hörmander [8].

Regulation and Passage to the limit

Let, $f, g \in \mathcal{D}'(\mathrm{R})$ the convulations $f_n = f * d_n$ and $g_n = g * d_n$ always exists as infinitely smooth functions for each n if (d_n) are delta sequences. Hence the elementary Schwartz products

$$f(g * \delta_n), (f * \delta_n)g, (f * \delta_n(g * \delta_n).$$

Then this suggest that there are four possibilities to define a product of f and g on using the delta sequences:

$$f.(g._n) = \lim_{n\to\infty} f(g * \delta_n), \tag{1}$$

$$f_n.g = \lim_{n\to\infty}(f * \delta_n)g, \tag{2}$$

$$(f_n).(g_n) = \lim_{n\to\infty}(f * \delta_n)(g * \varepsilon_n), \tag{3}$$

$$(f.g)_n = \lim_{n\to\infty}(f * \delta_n)(g * \delta_n), \tag{4}$$

Provided that the limits exist in $\mathcal{D}'$ (R). Equations (3) is due to Mikusinski [13], equations (1) and (2) to Hirata and Ogata [7], required both simultaneously, the equation (4) is due Antosik, Mikusinski and Skorski [1].

Later in [18], Temple followed the general idea of Mikunsinski and Sikorski in [14], and developed a sequential theory of distribution which are defined as "regular sequences" of arbitrary functions that can be used to define product of distributions and special functions, see [4].

We note that the above definitions are independent of the choice of the sequence, see Fisher [3]. In this study, we prove that the products are equal if they satisfy "a property" which we call semigroup condition then they satisfy the associativity property.

Although it seems difficult to multiply arbitrary distributions, it is usually possible to define the product of a distribution f and an infinitely differential function g and this is given in the next definition.

Definition 2 *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. Then product $fg = gf$ is defined by*

$$\langle f, g, \varphi\rangle = \langle gf, \varphi\rangle = \langle f, g\varphi\rangle$$

for all φ in $\mathcal{D}$.

It then follows easily by induction that

$$f^{(r)}g = \sum_{i=0}^{r}\binom{r}{i}(-1)^i[Fg^{(i)}]^{(r-i)}$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}$$

for $r = 1, 2\ldots$

Notice that if the product of two distributions $f, g \in \mathcal{D}'$ (R) exists then derivative of the product also exists and it can easily be verified that the product law

$$(f, g)' = f'g + fg'$$

for all $f, g \in \mathcal{D}'$ is valid.

The following definition was proposed by Fisher. It is a generalization of the definition 1, see [3].

Definition 3 *Let $f(x,r)$ and $g(x,r)$ be distributions in $\mathcal{D}$ and let*

$$f(x,r)_n = (f*\delta_n)(x), g(x,r)_n = (g*\delta_n)(x).$$

We say that the product $[f].[g]$ of $f(x,r)$ and $g(x,r)$ exists and is equal to h on the interval (a,b) if $\lim_{n\to\infty}\langle f(x,r)_n g(x,r)_n, \varphi(x)\rangle$ for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

The product thus defined is clearly commutative if it exists. The following result shows that the product $f.\,g$ exist but the product fg defined in **definition 2** does not exist.

$$x^{-r}.\delta^{(r-1)}(x) = \frac{(-1)^r r!}{(2r)!}\delta^{(2r-1)}(x) \tag{5}$$

Later Fisher gave the following non-commutative definition of the product, see [12].

Definition 4 *Let $f(x,r)$ and $g(x,r)$ be distributions in $\mathcal{D}'$ and let $g(x,r)_n = (g*\delta_n)(x)$. We say that the product $f.\,[g]$ of f and g exists and is equal to h on the interval (a, b) if*

$$\lim_{n\to\infty}\langle f(x,r)g(x,r)_n, \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b).

It is obvious that if the product fg exists by **Definition 2** then the product $[f].[g]$ exists by **Definition 3** is always commutative, the product defined in **Definition 4** is in general non-commutative. Hence in general $f.\,[g] \neq [f].\,g$ and the results obtained for these products contain constants which depend on the choice of the function ρ.

In general the two products above need not necessarily be equal. Several examples were given in [10], [11] and [12] that two products differ. Now we give in the following a new definition that these two products are equal under certain conditions.

Definition 5 *Let* $f(x,r)$ *and* $f(x,s)$ *be distributions in* $\mathcal{D}'$ *and let* $f(x,r)_n = (f * \delta_n)(t)$. *We say that the product is a semigroup product* $[f(x,r)].[f(x,s)]$ *(or* $f(x,r).[f(x,s)]$*) if* $f(x,r)$ *and* $f(x,s)$ *exist and is equal to* $f(x,r+s)$ *on the interval (a, b) if*

$$\lim_{n\to\infty}\langle f(x,r)_n f(x,s)_n, \varphi(x)\rangle = \langle f(x,r+s), \varphi(x)\rangle$$

or

$$\lim_{n\to\infty}\langle f(x,r) f(x,s)_n, \varphi(x)\rangle = \langle f(x,r+s), \varphi(x)\rangle$$

for all ϕ *in* $\mathcal{D}$ *with support contained in the interval* (a,b).

Theorem 1 If the products satisfy the semigroup property then two products are equal.

Proof: Let the products satisfy the semigroup condition. Then it can be easily seen on using the definitions that

$$\lim_{n\to\infty}\langle f(x,r) f(x,s)_n, \varphi(x)\rangle = \langle f(x,r+s), \varphi(x)\rangle$$

and

$$\lim_{n\to\infty}\langle f(x,r)_n f(x,s)_n, \varphi(x)\rangle = \langle f(x,r+s), \varphi(x)\rangle$$

exists respectively for all ϕ in $\mathcal{D}$. Then it follows that

$$\lim_{n\to\infty}(f(x,r) f(x,s)_n) = f(x,r+s)$$
$$\lim_{n\to\infty}(f(x,r)_n f(x,s)_n)$$

for all ϕ in $\mathcal{D}$. Then we have

$$\lim_{n\to\infty}\langle [f(x,r) - f(x,r)_n] f(x,s)_n, \varphi(x)\rangle = 0.$$

This proves the theorem.

Theorem 2 *The product of distributions is not always associative.*

Proof: Since

$$\langle x^{-1}x, \varphi\rangle = \int_0^\infty x^{-1}[x\varphi(x) + x\varphi(-x)]dx$$
$$= \int_0^\infty [\varphi(x) + \varphi(-x)]dx$$
$$= \int_{-\infty}^\infty \varphi(x)dx = \langle 1, \varphi\rangle$$

for all φ *in* $\mathcal{D}$ *and so* $x^{-1}x = 1$.

Thus, $(x^{-1}x)\delta = 1\delta = \delta$ but, $x^{-1}(x\delta) = x^{-1}0 = 0$.

Corollary 1 *If the product is a semigroup product then it also satisfies the associativity property.*

Proof: We will only prove the commutative case, non-commutative case can also be proved similarly. Let the product be a semigroup product and let $f(x,r), f(x,s)$ and $f(x,t)$ be distributions in $\mathcal{D}'$. On using the definition we write that

$$[f(x,s).f(x,t)] = \lim_{n\to\infty}(f(x,s)_n f(x,t)_n)$$
$$= f(x,t+s), \text{ then}$$

$$[f(x,s)].[f(x,t+s)] = f(x,r+s+t)$$

exists. On the order side we have that

$$[f(x,s).f(x,s)] = \lim_{n\to\infty}(f(x,r)_n f(x,s)_n)$$
$$= f(x,r+s)$$

and similarly

$$[f(x,r+s).f(x,t)] = f(x,r+s+t).$$

We can easily provide some counter examples that the converse of the above statement, in general, is not correct. For example, it was proved in [5], [12] respectively that the products $\delta^{(s)}.[\delta^{(r)}]$ and $[\delta^{(s)}].[\delta^{(r)}]$ exist and

$$[\delta^{(s)}].[\delta^{(r)}] = \delta^{(s)}.[\delta^{(r)}] = 0 \qquad (6)$$

for $s,r = 0, 1, 2,\ldots$ However it is obvious that the *product is not a semigroup product.*

Similarly, the distribution $(x+i0)^{-s}$ is defined as follows (see Gel'fand and Shilov [6])

$$(x+i0)^{-s} = x^{-s} + \frac{(-1)^s i\pi}{(s-1)!}\delta^{(s-1)}(x),$$

then it was proved in [5] that the product is a semigroup since

$$[(x+i0)^{-s}].(x+i0)^{-r} = (x+i0)^{-r-s}$$
$$= [(x+i0)^{-s}].[(x+i0)^{-r}].$$

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