UNIVERSITI PUTRA MALAYSIA

RUNGE-KUTTA-NYSTROM METHODS FOR SOLVING
OSCILLATORY PROBLEMS

NORAZAK BIN SENU
FS 2010 23
RUNGE-KUTTA-NYSTROM METHODS FOR SOLVING OSCILLATORY PROBLEMS

NORAZAK BIN SENU

DOCTOR OF PHILOSOPHY
UNIVERSITI PUTRA MALAYSIA

2010
RUNGE-KUTTA-NYSTRÖM METHODS FOR SOLVING OSCILLATORY PROBLEMS

By

NORAZAK BIN SENU

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

February 2010
New Runge-Kutta-Nyström (RKN) methods are derived for solving system of second-order Ordinary Differential Equations (ODEs) in which the solutions are in the oscillatory form. The dispersion and dissipation relations are imposed to get methods with the highest possible order of dispersion and dissipation. The derivation of Embedded Explicit RKN (ERKN) methods for variable step size codes are also given. The strategies in choosing the free parameters are also discussed. We analyze the numerical behavior of the RKN and ERKN methods both theoretically and experimentally and comparisons are made over the existing methods.

In the second part of this thesis, a Block Embedded Explicit RKN (BERKN) method are developed. The implementation of BERKN method is discussed. The numerical results are compared with non block method. We find that the new code on Block Embedded Explicit RKN (BERKN) method is more efficient for solving system of second-order ODEs directly.

Next, we discussed the derivation of Diagonally Implicit RKN (DIRKN) methods for solving stiff second order ODEs in which the solutions are oscillating functions. The dispersion and
dissipation relations are developed and again are imposed in the derivation of the methods. For solving oscillatory problems with high frequency, method with P-stability property is discussed. We also derive the Embedded Diagonally Implicit RKN (EDIRKN) methods for variable step size codes. To see the preciseness and effectiveness of the methods, the constant and variable step size codes are developed and numerical results are compared with current methods given in the literature.

Finally, the Parallel Embedded Explicit RKN (PERKN) method is developed. The parallel implementation of PERKN on the parallel machine is discussed. The performance of the PERKN algorithm for solving large system of ODEs are presented. We observe that the PERKN gives the better performance when solving large system of ODEs.

In conclusion, the new codes developed in this thesis are suitable for solving system of second-order ODEs in which the solutions are in the oscillatory form.
Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH RUNGE-KUTTA-NYSTROM BAGI MENYELESAIKAN MASALAH BERAYUNAN

Oleh

NORAZAK BIN SENU

Februari 2010

Pengerusi : Professor Dato’ Mohamed bin Suleiman, PhD
Fakulti : Sains


Kesimpulannya, kod baharu yang dibangunkan di dalam tesis ini sesuai untuk sistem PPB peringkat dua yang mana penyelesaian adalah dalam bentuk berayunan.
ACKNOWLEDGEMENTS

In the Name of Allah the Most Compassionate, the Most Merciful First and foremost

First all, praise is for Allah Subhanahu Wa Taala for giving me the strength, guidance and patience to complete this thesis. May blessing and peace be upon Prophet Muhammad Sallalahu Alaihi Wasallam, who was sent for mercy to the world.

I wish to express my sincere and deepest gratitude to the chairman of the supervisory committee, YBhg. Professor Dato’ Dr. Mohamed bin Suleiman for his invaluable advice, guidance, assistance and most of all, for his constructive criticisms. This work would not have been completed without his help that I received in various aspects of the research.

I am also grateful to the member of the supervisory committee, Associate Professor Dr. Fudziah bt Ismail and Professor Dr. Mohamed bin Othman. I also wish to express my thanks to all of my friends during my study in Universiti Putra Malaysia. I would like to thank all staffs of the Department of Mathematics. Their continuous help, encouragement and support are highly appreciated. I thank my employer, Universiti Putra Malaysia for providing me with the UPM scholarship which funded this research during most of my studies and also who granted me study leave.

Finally, I cannot put into words how much I appreciate the continuous support, understanding and patience of my wife, Norfifah, and my children, Nor Fatin Aqilah, Muhammad Farhan Aqil and Muhammad Fath Hadif and special thanks to my mother Hjh. Jamenah bt. Sirat and my father Hj. Senu bin Sabikan for their continuous encouragement. Thank you.
I certify that a Thesis Examination Committee has met on 22 February 2010 to conduct
the final examination of Norazak bin Senu on his thesis entitled “Runge-Kutta-Nyström
Methods for Solving Oscillatory Problems” in accordance with Universities and Univer-
sity Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doc-
tor of Philosophy.

Members of the Thesis Examination Committee were as follows:

Norihan Md. Arifin, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Malik Hj Abu Hassan, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Leong Wah Jun, PhD
Lecturer
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Bachok M. Taib, PhD
Professor
Faculty of Science
Universiti Sains Islam Malaysia
(External Examiner)

BUJANG BIN KIM HUAT, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 15 April 2010
This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

**Dato’ Mohamed Suleiman, PhD**  
Professor  
Faculty of Science  
Universiti Putra Malaysia  
(Chairman)

**Fudziah Ismail, PhD**  
Associate Professor  
Faculty of Science  
Universiti Putra Malaysia  
(Member)

**Mohamed Othman, PhD**  
Professor  
Faculty of Computer Science and Information Technology  
Universiti Putra Malaysia  
(Member)

---

HASANAH MOHD GHAZALI, PhD  
Professor and Dean  
School of Graduate Studies  
Universiti Putra Malaysia

Date: 13 May 2010
DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

NORAZAK BIN SENU

Date: 22 February 2010
TABLE OF CONTENTS

ABSTRACT ii
ABSTRAK iv
ACKNOWLEDGEMENTS vi
APPROVAL vii
DECLARATION ix
LIST OF TABLES xiii
LIST OF FIGURES xx
LIST OF ABBREVIATIONS xxiv

CHAPTER

1 INTRODUCTION 1
1.1 Literature Review 1
1.2 The Objectives of the Thesis 4
1.3 Outline of the Thesis 5
1.4 The Initial Value Problem 6
1.5 Runge-Kutta-Nyström Method 7
1.6 Algebraic Conditions for RKN Method 9
1.7 Local Truncation Error 13
1.8 Analysis of the Periodicity and Absolute Stability 15
1.9 Analysis of Dispersion (Phase-lag) and Dissipation 23
1.10 The Stiff Problem 28

2 AN EXPLICIT RUNGE-KUTTA-NYSTRÖM (RKN) METHODS FOR SOLVING OSCILLATORY PROBLEMS 30
2.1 Introduction 30
2.2 Derivation of Three-stage Third-order RKN Methods 33
  2.2.1 Problems Tested 41
  2.2.2 Numerical Results 43
  2.2.3 Discussion 55
2.3 Derivation of Four-stage Fourth-order RKN Methods 56
  2.3.1 Numerical Results 63
  2.3.2 Discussion 75
2.4 Derivation of Four-stage Fifth-order RKN Methods 76
  2.4.1 Numerical Results 83
  2.4.2 Discussion 94

3 AN EMBEDDED EXPLICIT RUNGE-KUTTA-NYSTRÖM METHODS (ERKN) FOR SOLVING OSCILLATORY PROBLEMS 95
3.1 Introduction 95
3.2 Derivation of Three-stage Embedded RKN Methods 97
  3.2.1 Derivation of 3(2) Pair RKN Methods 97
  3.2.2 Estimating the Error and Step Size Selection 102
6.3 Derivation of Embedded DIRKN Method with P-stability Property 254
6.3.1 Numerical Results 257
6.3.2 Discussion 263

7 PARALLEL EMBEDDED EXPLICIT RUNGE-KUTTA-NYSTRÖM (PERKN) METHOD FOR SOLVING SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS 264
7.1 Introduction 264
7.2 Parallel Programming 266
  7.2.1 High Performance Computer Sunfire 1280 Architecture 266
  7.2.2 Message Passing Interface (MPI) 268
  7.2.3 Performance of Parallel Algorithm 268
7.3 Derivation of Parallel Embedded Explicit RKN (PERKN) Method 271
7.4 Implementation of PERKN5(4) Method on Parallel Machines 275
7.5 Problem Tested 277
7.6 Numerical Results 279
7.7 Discussion 284

8 CONCLUSION 286
8.1 Summary 286
8.2 Future Work 287

BIBLIOGRAPHY 289
BIODATA OF STUDENT 298
LIST OF PUBLICATIONS 299
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The RKN3(3,6,3)M method</td>
</tr>
<tr>
<td>2.2</td>
<td>The RKN3(3,6,3)S method</td>
</tr>
<tr>
<td>2.3</td>
<td>The RKN3(3,6,5) method</td>
</tr>
<tr>
<td>2.4</td>
<td>The RKN3(3,6,1) method</td>
</tr>
<tr>
<td>2.5</td>
<td>Summary of the characteristic of the third-order explicit RKN methods</td>
</tr>
<tr>
<td>2.6</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.1</td>
</tr>
<tr>
<td>2.7</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.2</td>
</tr>
<tr>
<td>2.8</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.3</td>
</tr>
<tr>
<td>2.9</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.4</td>
</tr>
<tr>
<td>2.10</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.5</td>
</tr>
<tr>
<td>2.11</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.6</td>
</tr>
<tr>
<td>2.12</td>
<td>Comparison results between RKN3(3,6,∞), RKN3(3,6,∞)HS, RKN3(3,6,5), RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS and RKN3(3,12,3)HS methods when solving Problem 2.7</td>
</tr>
</tbody>
</table>
2.13 The RKN4(4,8,5)M method 58
2.14 The RKN4(4,8,5)S method 59
2.15 The RKN4(4,8,7) method 61
2.16 Summary of the characteristic of the fourth-order RKN methods 63
2.17 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.1 65
2.18 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.2 66
2.19 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.3 67
2.20 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.4 68
2.21 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.5 69
2.22 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.6 70
2.23 Comparison results between RKN4(4,8,7), RKN4(4,8,5)M, RKN4(4,8,5)S, RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.7 71
2.24 The RKN4(5,8,5)M method 77
2.25 The RKN4(5,8,5)S method 78
2.26 The RKN4(5,8,7) method 81
2.27 Summary of the characteristic of the fifth-order RKN methods 82
2.28 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.1 84
2.29 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.2 85

xiv
2.30 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.3

2.31 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.4

2.32 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.5

2.33 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.6

2.34 Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.7

3.1 The ERKN3(2)M method

3.2 The ERKN3(2)S method

3.3 The ERKN3(2)D5 method

3.4 The ERKN3(2)Z method

3.5 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.1

3.6 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.2

3.7 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.3

3.8 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.4

3.9 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.5

3.10 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.6

3.11 Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.7

3.12 The ERKN4(3)M method

xv
3.13 The ERKN4(3)S method

3.14 The ERKN4(3)D7 method

3.15 Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.1

3.16 Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.2

3.17 Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.3

3.18 Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.4


3.20 Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.6

3.21 Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.7

3.22 The ERKN5(4)M method

3.23 The ERKN5(4)S method

3.24 The ERKN5(4)D7 method

3.25 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.1

3.26 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.2

3.27 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.3

3.28 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.4

3.29 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.5

3.30 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.6

3.31 Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D,
4.1 The ERKN3(2) method for $y_{n+2}$ and $y'_{n+2}$ of Type A 160
4.2 The Block Embedded Explicit RKN (BERKN3(2)TA) of Type A method 161
4.3 The ERKN3(2) method for $y_{n+2}$ and $y'_{n+2}$ of Type B 173
4.4 The Block Embedded Explicit RKN (BERKN3(2)TB) of Type B method 173
4.5 Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M, RK3(2)D and RK4(3)F for Problem 2.1 176
4.6 Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M, RK3(2)D and RK4(3)F for Problem 2.3 177
4.7 Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M, RK3(2)D and RK4(3)F for Problem 2.4 178
4.8 Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M, RK3(2)D and RK4(3)F for Problem 2.7 179
4.9 Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M, RK3(2)D and RK4(3)F for Problem 2.8 180
5.1 The DIRKN3(4,4)(a) method 198
5.2 The DIRKN3(4,4)(b) method 198
5.3 The DIRKN3(4,6)(a) method 200
5.4 The DIRKN3(4,6)(b) method 201
5.5 Summary of the characteristic of the three-stage fourth-order DIRKN methods 202
5.6 Comparing our results with the methods in the literature for Problem 5.1 207
5.7 Comparing our results with the methods in the literature for Problem 5.2 208
5.8 Comparing our results with the methods in the literature for Problem 5.3 209
5.9 Comparing our results with the methods in the literature for Problem 5.4 210
5.10 The DIRKN4(4,4)a method 217
5.11 The DIRKN4(4,4)b method 218
5.12 The DIRKN4(4,8) method 223
5.13 The DIRKN4(4,4)P method 227
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.10</td>
<td>Comparison numerical results between EDIRKN4(3)P and DIRK4(3)B for Problem 5.7</td>
<td>259</td>
</tr>
<tr>
<td>6.11</td>
<td>Comparison numerical results between EDIRKN4(3)P and DIRK4(3)B for Problem 5.8</td>
<td>260</td>
</tr>
<tr>
<td>7.1</td>
<td>Hardware Configuration of Sunfire 1280</td>
<td>267</td>
</tr>
<tr>
<td>7.2</td>
<td>Runge-Kutta matrix suggested by Burrage (1990)</td>
<td>272</td>
</tr>
<tr>
<td>7.3</td>
<td>A new Runge-Kutta matrix adapted for PERKN method</td>
<td>272</td>
</tr>
<tr>
<td>7.4</td>
<td>The Parallel Embedded Explicit RKN (PERKN5(4)) method</td>
<td>274</td>
</tr>
<tr>
<td>7.5</td>
<td>Numerical results of sequential and parallel implementation of PERKN5(4) for Problem 7.1</td>
<td>280</td>
</tr>
<tr>
<td>7.6</td>
<td>Numerical results of sequential and parallel implementation of PERKN5(4) for Problem 7.2</td>
<td>282</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Stability region for RKN3(3,6,3)M method with ( m = 3; p = 3; q = 6 ) and ( r = 3 )</td>
<td>35</td>
</tr>
<tr>
<td>2.2</td>
<td>Stability region for RKN3(3,6,3)S method with ( m = 3; p = 3; q = 6 ) and ( r = 3 )</td>
<td>36</td>
</tr>
<tr>
<td>2.3</td>
<td>Stability region for RKN3(3,6,5) method with ( m = 3; p = 3; q = 6 ) and ( r = 5 )</td>
<td>38</td>
</tr>
<tr>
<td>2.4</td>
<td>Histogram of accuracy for third-order RKN methods for Problem 2.1 with ( h = 0.025 )</td>
<td>52</td>
</tr>
<tr>
<td>2.5</td>
<td>Histogram of accuracy for third-order RKN methods for Problem 2.2 with ( h = 0.025 )</td>
<td>52</td>
</tr>
<tr>
<td>2.6</td>
<td>Histogram of accuracy for third-order RKN methods for Problem 2.3 with ( h = 0.025 )</td>
<td>53</td>
</tr>
<tr>
<td>2.7</td>
<td>Histogram of accuracy for third-order RKN methods for Problem 2.4 with ( h = 0.025 )</td>
<td>53</td>
</tr>
<tr>
<td>2.8</td>
<td>Histogram of accuracy for third-order RKN methods for Problem 2.5 with ( h = 0.025 )</td>
<td>54</td>
</tr>
<tr>
<td>2.9</td>
<td>Histogram of accuracy for third-order RKN methods for Problem 2.6 with ( h = 0.025 )</td>
<td>54</td>
</tr>
<tr>
<td>2.10</td>
<td>Stability region for RKN4(4,8,5)S method with ( m = 4; p = 4; ) and ( q = 8 )</td>
<td>60</td>
</tr>
<tr>
<td>2.11</td>
<td>Stability region for RKN4(4,8,5)S method for the range (-2 \leq \text{Re}(H) \leq 0)</td>
<td>60</td>
</tr>
<tr>
<td>2.12</td>
<td>Stability region for RKN4(4,8,7) method</td>
<td>62</td>
</tr>
<tr>
<td>2.13</td>
<td>Histogram of accuracy for fourth-order RKN methods for Problem 2.1 with ( h = 0.025 )</td>
<td>72</td>
</tr>
<tr>
<td>2.14</td>
<td>Histogram of accuracy for fourth-order RKN methods for Problem 2.2 with ( h = 0.025 )</td>
<td>72</td>
</tr>
<tr>
<td>2.15</td>
<td>Histogram of accuracy for fourth-order RKN methods for Problem 2.3 with ( h = 0.025 )</td>
<td>73</td>
</tr>
<tr>
<td>2.16</td>
<td>Histogram of accuracy for fourth-order RKN methods for Problem 2.4 with ( h = 0.025 )</td>
<td>73</td>
</tr>
<tr>
<td>2.17</td>
<td>Histogram of accuracy for fourth-order RKN methods for Problem 2.5 with ( h = 0.05 )</td>
<td>74</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>2.18</td>
<td>Histogram of accuracy for fourth-order RKN methods for Problem 2.6 with $h = 0.05$</td>
<td>74</td>
</tr>
<tr>
<td>2.19</td>
<td>Stability region for RKN4($5,8,5$)S method</td>
<td>78</td>
</tr>
<tr>
<td>2.20</td>
<td>Stability region for RKN4($5,8,5$)S method for the range $-2.5 \leq \text{Re}(H) \leq 0$</td>
<td>79</td>
</tr>
<tr>
<td>2.21</td>
<td>Stability region for RKN4($5,8,7$) method</td>
<td>80</td>
</tr>
<tr>
<td>2.22</td>
<td>Stability region for RKN4($5,8,7$) method for the range $-2.5 \leq \text{Re}(H) \leq 0$</td>
<td>82</td>
</tr>
<tr>
<td>2.23</td>
<td>Histogram of accuracy for fifth-order RKN methods for Problem 2.1 with $h = 0.025$</td>
<td>91</td>
</tr>
<tr>
<td>2.24</td>
<td>Histogram of accuracy for fifth-order RKN methods for Problem 2.2 with $h = 0.025$</td>
<td>91</td>
</tr>
<tr>
<td>2.25</td>
<td>Histogram of accuracy for fifth-order RKN methods for Problem 2.3 with $h = 0.05$</td>
<td>92</td>
</tr>
<tr>
<td>2.26</td>
<td>Histogram of accuracy for fifth-order RKN methods for Problem 2.4 with $h = 0.05$</td>
<td>92</td>
</tr>
<tr>
<td>2.27</td>
<td>Histogram of accuracy for fifth-order RKN methods for Problem 2.5 with $h = 0.1$</td>
<td>93</td>
</tr>
<tr>
<td>2.28</td>
<td>Histogram of accuracy for fifth-order RKN methods for Problem 2.6 with $h = 0.05$</td>
<td>93</td>
</tr>
<tr>
<td>3.1</td>
<td>Efficiency curve for 3(2) pair of RKN methods for Problem 2.1</td>
<td>112</td>
</tr>
<tr>
<td>3.2</td>
<td>Efficiency curve for 3(2) pair of RKN methods for Problem 2.2</td>
<td>112</td>
</tr>
<tr>
<td>3.3</td>
<td>Efficiency curve for 3(2) pair of RKN methods for Problem 2.3</td>
<td>113</td>
</tr>
<tr>
<td>3.4</td>
<td>Efficiency curve for 3(2) pair of RKN methods for Problem 2.6</td>
<td>113</td>
</tr>
<tr>
<td>3.5</td>
<td>Efficiency curve for 4(3) pair of RKN methods for Problem 2.1</td>
<td>128</td>
</tr>
<tr>
<td>3.6</td>
<td>Efficiency curve for 4(3) pair of RKN methods for Problem 2.2</td>
<td>128</td>
</tr>
<tr>
<td>3.7</td>
<td>Efficiency curve for 4(3) pair of RKN methods for Problem 2.5</td>
<td>129</td>
</tr>
<tr>
<td>3.8</td>
<td>Efficiency curve for 4(3) pair of RKN methods for Problem 2.7</td>
<td>129</td>
</tr>
<tr>
<td>3.9</td>
<td>Efficiency curve for 5(4) pair of RKN methods for Problem 2.1</td>
<td>144</td>
</tr>
<tr>
<td>3.10</td>
<td>Efficiency curve for 5(4) pair of RKN methods for Problem 2.2</td>
<td>144</td>
</tr>
<tr>
<td>3.11</td>
<td>Efficiency curve for 5(4) pair of RKN methods for Problem 2.3</td>
<td>145</td>
</tr>
</tbody>
</table>

xxi
3.12 Efficiency curve for 5(4) pair of RKN methods for Problem 2.6 145

4.1 Stability region for Block Explicit RKN method of Type A 161

4.2 Stability region for Block Explicit RKN method of Type B 172

4.3 Efficiency curve for block methods for Problem 2.1 181

4.4 Efficiency curve for block methods for Problem 2.3 181

4.5 Efficiency curve for block methods for Problem 2.4 182

4.6 Efficiency curve for block methods for Problem 2.7 182

4.7 Efficiency curve for block methods for Problem 2.8 183

5.1 Stability region for DIRKN3(4,6)(a) method 201

5.2 Stability region for DIRKN3(4,6)(b) method 202

5.3 Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.1 with $h = 0.01$ 211

5.4 Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.2 method with $h = 0.25$ 211

5.5 Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.3 method with $h = 0.0625$ 212

5.6 Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.4 method with $h = 0.01$ 212

5.7 Stability region for DIRKN4(4,8) method 224

5.8 Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.1 with $h = 0.01$ 233

5.9 Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.2 with $h = 0.0625$ 233

5.10 Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.3 with $h = 0.25$ 234

5.11 Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.4 with $h = 0.01$ 234

5.12 Histogram of accuracy for P-stable four-stage forth-order DIRKN method for Problem 5.5 with $h = 0.0025$ 237

5.13 Histogram of accuracy for P-stable four-stage forth-order DIRKN method for Problem 5.6 with $h = 0.0625$ 237
6.1 Efficiency curve for 4(3) pair DIRKN methods for Problem 5.1 251
6.2 Efficiency curve for 4(3) pair DIRKN methods for Problem 5.2 251
6.3 Efficiency curve for 4(3) pair DIRKN methods for Problem 5.3 252
6.4 Efficiency curve for 4(3) pair DIRKN methods for Problem 5.4 252
6.5 Stability region for embedded formula of DIRKN(3)P 256
6.6 Efficiency curve for DIRKN(3)P method for Problem 5.5 261
6.7 Efficiency curve for DIRKN(3)P method for Problem 5.6 261
6.8 Efficiency curve for DIRKN(3)P method for Problem 5.7 262
6.9 Efficiency curve for DIRKN(3)P method for Problem 5.8 262
7.1 MPI program structure 269
7.2 Speedup for PERKN5(4) method on Parallel Machine when solving Problem 7.1 281
7.3 Efficiency for PERKN5(4) method on Parallel Machine when solving Problem 7.1 281
7.4 Speedup for PERKN5(4) method on Parallel Machine when solving Problem 7.2 283
7.5 Efficiency for PERKN5(4) method on Parallel Machine when solving Problem 7.2 283

xxiii